Note on Classical Mechanics

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Part I
Review

1 Newtonian Mechanics

1.1 Coordinate System

- **Polar**

$$\begin{cases} \boldsymbol{e}_r = \boldsymbol{e}_x \cos\theta + \boldsymbol{e}_y \sin\theta \\ \boldsymbol{e}_\theta = -\boldsymbol{e}_x \sin\theta + \boldsymbol{e}_y \cos\theta \end{cases} \tag{1.1}$$

$$\begin{cases} \dot{\boldsymbol{e}}_r = -\boldsymbol{e}_x \dot\theta \sin\theta + \boldsymbol{e}_y \dot\theta \cos\theta \\ \dot{\boldsymbol{e}}_\theta = -\boldsymbol{e}_x \dot\theta \cos\theta - \boldsymbol{e}_y \dot\theta \sin\theta \end{cases} \tag{1.2}$$

$$\begin{cases} \ddot{\boldsymbol{e}}_r = -\boldsymbol{e}_x \left(\ddot\theta \sin\theta + \dot\theta^2 \cos\theta\right) + \boldsymbol{e}_y \left(\ddot\theta \cos\theta - \dot\theta^2 \sin\theta\right) = -\boldsymbol{e}_r \dot\theta^2 + \boldsymbol{e}_\theta \ddot\theta \\ \ddot{\boldsymbol{e}}_\theta = \boldsymbol{e}_x \left(-\ddot\theta \cos\theta + \dot\theta^2 \sin\theta\right) - \boldsymbol{e}_y \left(\ddot\theta \sin\theta + \dot\theta^2 \cos\theta\right) = -\boldsymbol{e}_r \ddot\theta + \boldsymbol{e}_\theta \dot\theta^2 \end{cases} \tag{1.3}$$

- **Spherical**

$$\begin{cases} \boldsymbol{e}_r = \boldsymbol{e}_x \sin\theta\cos\varphi + \boldsymbol{e}_y \sin\theta\sin\varphi + \boldsymbol{e}_z \cos\theta \\ \boldsymbol{e}_\theta = \boldsymbol{e}_x \cos\theta\cos\varphi + \boldsymbol{e}_y \cos\theta\sin\varphi - \boldsymbol{e}_z \sin\theta \\ \boldsymbol{e}_\varphi = -\boldsymbol{e}_x \sin\varphi + \boldsymbol{e}_y \cos\varphi \end{cases} \tag{1.4}$$

$$\begin{cases} \dot{\boldsymbol{e}}_r = \boldsymbol{e}_x \left(\dot\theta \cos\theta\cos\varphi - \dot\varphi \sin\theta\sin\varphi\right) + \boldsymbol{e}_y \left(\dot\theta \cos\theta\sin\varphi + \dot\varphi \sin\theta\cos\varphi\right) - \boldsymbol{e}_z \dot\theta \sin\theta = \boldsymbol{e}_\theta \dot\theta + \boldsymbol{e}_\varphi \dot\varphi \sin\theta \\ \dot{\boldsymbol{e}}_\theta = -\boldsymbol{e}_x \left(\dot\theta \sin\theta\cos\varphi + \dot\varphi \cos\theta\sin\varphi\right) + \boldsymbol{e}_y \left(-\dot\theta \sin\theta\sin\varphi + \dot\varphi \cos\theta\cos\varphi\right) - \boldsymbol{e}_z \dot\theta \cos\theta = -\boldsymbol{e}_r \dot\theta + \boldsymbol{e}_\varphi \dot\varphi \cos\theta \\ \dot{\boldsymbol{e}}_\varphi = -\boldsymbol{e}_x \dot\varphi \cos\varphi - \boldsymbol{e}_y \dot\varphi \sin\varphi = -\boldsymbol{e}_r \dot\varphi \sin\theta - \boldsymbol{e}_\theta \dot\varphi \cos\theta \end{cases} \tag{1.5}$$

$$\begin{cases} \ddot{\boldsymbol{e}}_r = \boldsymbol{e}_\theta \ddot\theta + \boldsymbol{e}_\varphi \left(\ddot\varphi \sin\theta + \dot\theta\dot\varphi \cos\theta\right) \\ \ddot{\boldsymbol{e}}_\theta = -\boldsymbol{e}_r \ddot\theta + \boldsymbol{e}_\varphi \left(\ddot\varphi \cos\theta - \dot\theta\dot\varphi \sin\theta\right) \\ \ddot{\boldsymbol{e}}_\varphi = -\boldsymbol{e}_r \left(\ddot\varphi \sin\theta + \dot\theta\dot\varphi \cos\theta\right) - \boldsymbol{e}_\theta \left(\ddot\varphi \cos\theta - \dot\theta\dot\varphi \sin\theta\right) \end{cases} \tag{1.6}$$

1.2 Dynamics

Polar $\begin{cases} m\left(\ddot r - r\dot\theta^2\right) = F_r \\ m\left(r\ddot\theta + 2\dot r\dot\theta\right) = F_\theta \end{cases}$

Spherical $\begin{cases} m\left(\ddot r - r\dot\theta^2 - r\dot\varphi^2 \sin^2\theta\right) = F_r \\ m\left(r\ddot\theta + 2\dot r\dot\theta - r\dot\varphi^2 \sin\theta\cos\theta\right) = F_\theta \\ m\left(r\ddot\varphi \sin\theta + 2\dot r\dot\varphi \sin\theta + 2r\dot\varphi\dot\theta \cos\theta\right) = F_\varphi \end{cases}$

Cylindrical $\begin{cases} m\left(\ddot R - R\dot\varphi^2\right) = F_R \\ m\left(R\ddot\varphi + 2\dot R\dot\varphi\right) = F_\varphi \\ m\ddot z = F_z \end{cases}$

Intrinsic $\begin{cases} m\frac{dv}{dt} = F_r \\ m\frac{v^2}{\rho} = F_n \end{cases}$

Part II

Analytical Theories

2 Lagrange's Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0 \tag{2.1}$$

Assumptions:

- Constraints are holonomic $\Rightarrow \boldsymbol{r} = \boldsymbol{r}(q_1, q_2, ..., q_\alpha, t)$
- Constraint forces do no work $\Rightarrow \sum_{i=1}^{n} \boldsymbol{N}_i \cdot \delta \boldsymbol{r}_i = 0$
- Applied forces are conservative $\Rightarrow \boldsymbol{F}_i = -\boldsymbol{\nabla}_i V$
- Potential V does not depend on $\dot{q} \Rightarrow \frac{\partial V}{\partial \dot{q}} = 0$

2.1 Derivation

2.1.1 From D'Alembert's Principle to Lagrange's Equations

D'Alembert's Principle

$$\sum_{i=1}^{n}(\boldsymbol{F}_i - m_i \ddot{\boldsymbol{r}}_i) \cdot \delta \boldsymbol{r}_i = 0 \tag{2.2}$$

- The first part

$$\begin{aligned}
\sum_{i=1}^{n} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i &= \sum_{i=1}^{n} \left(\boldsymbol{F}_i \cdot \sum_{\alpha=1}^{s} \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha} \delta q_\alpha \right) \\
&= \sum_{\alpha=1}^{s} \left(\sum_{i=1}^{n} \boldsymbol{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \right) \delta q_\alpha \\
&= \sum_{\alpha=1}^{s} Q_\alpha \delta q_\alpha
\end{aligned}$$

- The second part

$$\begin{aligned}
-\sum_{i=1}^{n} m_i \ddot{\boldsymbol{r}}_i \cdot \delta \boldsymbol{r}_i &= -\sum_{i=1}^{n} \left(m_i \ddot{\boldsymbol{r}}_i \cdot \sum_{\alpha=1}^{s} \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha} \delta q_\alpha \right) = -\sum_{\alpha=1}^{s} \left(\sum_{i=1}^{n} m_i \ddot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha} \right) \delta q_\alpha \\
&= -\sum_{\alpha=1}^{s} \left(\sum_{i=1}^{n} m_i \frac{\mathrm{d}\dot{\boldsymbol{r}}_i}{\mathrm{d}t} \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha} \right) \delta q_\alpha \\
&= -\sum_{\alpha=1}^{s} \left[\sum_{i=1}^{n} m_i \frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha} \right) - \sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \boldsymbol{r}_i}{\partial q_\alpha} \right) \right] \delta q_\alpha \\
&= -\sum_{\alpha=1}^{s} \left[\sum_{i=1}^{n} m_i \frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_\alpha} \right) - \sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_\alpha} \right] \delta q_\alpha \\
&= -\sum_{\alpha=1}^{s} \left[\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}_\alpha} \left(\sum_{i=1}^{n} \frac{1}{2} m_i \left| \dot{\boldsymbol{r}}_i \right|^2 \right) - \frac{\partial}{\partial q_\alpha} \left(\sum_{i=1}^{n} \frac{1}{2} m_i \left| \dot{\boldsymbol{r}}_i \right|^2 \right) \right] \delta q_\alpha \\
&= -\sum_{\alpha=1}^{s} \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} \right) \delta q_\alpha
\end{aligned}$$

Then we have

$$\sum_{\alpha=1}^{s} \left(Q_\alpha - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_\alpha} + \frac{\partial T}{\partial q_\alpha} \right) \delta q_\alpha = 0$$

Since the set of virtual displacement δq_α are independent, the only way for the equation above to hold is that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} = Q_\alpha \quad (\alpha = 1, 2, \cdots, s)$$

If we now limit ourselves to **conservative systems**, we must have

$$\boldsymbol{F}_i = -\boldsymbol{\nabla}_i V$$

and similarly,

$$Q_\alpha = \sum_{i=1}^{n} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha} = -\sum_{i=1}^{n} \boldsymbol{\nabla}_i V \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha} \quad (\alpha = 1, 2, \cdots, s)$$

We now define the **Lagrangian** for the system as

$$L = T - V$$

we can rewrite the equation above as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial (T - V)}{\partial \dot{q}_\alpha} - \frac{\partial (T - V)}{\partial q_\alpha} = 0$$

we finally obtain **Lagrange's equations**

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0 \quad (\alpha = 1, 2, \cdots, s)$$

2.1.2 From Hamilton's Principle to Lagrange's Equations

$$I = \int_{t_1}^{t_2} L \mathrm{d}t$$

$$\begin{aligned}
\delta I =& \delta \int_{t_1}^{t_2} L(q_\alpha, \dot{q}_\alpha, t) \mathrm{d}t \\
=& \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_\alpha} \delta q_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \delta \dot{q}_\alpha \right) \mathrm{d}t \\
=& \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_\alpha} \delta q_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \frac{\mathrm{d}}{\mathrm{d}t} (\delta q_\alpha) \right] \mathrm{d}t \\
=& \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_\alpha} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \right) \right] \delta q_\alpha \mathrm{d}t + \left[\frac{\partial L}{\partial \dot{q}_\alpha} \delta q_\alpha \right]_{t_1}^{t_2} \\
=& \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_\alpha} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \right) \right] \delta q_\alpha \mathrm{d}t = 0
\end{aligned}$$

Since the set of virtual displacement δq_α are independent, the only way for the equation above to hold is that

$$\frac{\partial L}{\partial q_\alpha} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \right) = 0 \quad (\alpha = 1, 2, \cdots, s)$$

2.2 Conservation Theorems

2.2.1 The Kinetic Energy

$$T = \frac{1}{2} \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha^2 \tag{2.3}$$

$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_j, t)$

$$\dot{\mathbf{r}}_\alpha = \sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_\alpha}{\partial t} \tag{2.4}$$

$$\dot{\mathbf{r}}_\alpha \dot{\mathbf{r}}_\alpha = \sum_{j,k} \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \frac{\partial \mathbf{r}_\alpha}{\partial q_k} \dot{q}_j \dot{q}_k + 2 \sum_j \frac{\partial \mathbf{r}_\alpha}{\partial q_j} \frac{\partial \mathbf{r}_\alpha}{\partial t} \dot{q}_j + \frac{\partial \mathbf{r}_\alpha}{\partial t} \frac{\partial \mathbf{r}_\alpha}{\partial t} \tag{2.5}$$

$$\sum_j \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j = 2T \tag{2.6}$$

2.2.2 Conservation of Energy

$$
\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t} &= \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t} \\
&= \sum_j \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t} \\
&= \sum_j \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) + \frac{\partial L}{\partial t}
\end{aligned} \tag{2.7}
$$

It therefore follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L\right) + \frac{\partial L}{\partial t} = \frac{\mathrm{d}H}{\mathrm{d}t} + \frac{\partial L}{\partial t} = 0 \tag{2.8}$$

Where we introduce a new function

$$H(q, \dot{q}, t) = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L \tag{2.9}$$

In cases where the Lagrangian is not explicitly dependent on time we find that

$$H(q, \dot{q}, t) = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = cste \tag{2.10}$$

Eq.(2.11) can be written as

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = 2T - L = T + U = E = cste \tag{2.11}$$

The function H is called the Hamiltonian of the system and it is equaled to the total energy only if the following conditions are met:

1. The equations of the transformation connecting the Cartesian and generalized coordinates must be independent of time.
2. The potential energy must be velocity independent.

2.2.3 Noether's Theorem: Invariantion → Conservation

$$\delta L = \sum_j \left(\frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j\right) = \sum_j \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j\right] = \frac{\mathrm{d}}{\mathrm{d}t}\sum_j \left(\frac{\partial L}{\partial \dot{q}_j}\delta q_j\right) \tag{2.12}$$

- **Conservation of Linear Momentum**

  If the generalized coordinate q_i is cyclic, then the corresponding generalized momentum component p_i to be a constant of motion.

  $$\text{Generalized Momentum} \quad p_i = \frac{\partial L}{\partial \dot{q}_i} = cste$$

- **Conservation of Angular Momentum**

  $$\mathbf{L} = \sum_\alpha (\mathbf{r}_\alpha \times \mathbf{p}_\alpha) = cste$$

- **Conservation of Hamiltonian**

3 Euler's Equation

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = 0 \tag{3.1}$$

3.1 Techniques of the Calculus of Differential

$$J = \int_a^b F(x, y, y')\mathrm{d}x$$

$$y(x) = y_0(x) + \alpha\eta(x)$$

$$J(\alpha) = \int_a^b F(x, y_0 + \alpha\eta, y_0' + \alpha\eta')\mathrm{d}x$$

$$\begin{aligned}
\frac{\mathrm{d}J}{\mathrm{d}\alpha} &= \int_a^b \left(\frac{\partial F}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha}\right)\mathrm{d}x \\
&= \int_a^b \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)\mathrm{d}x \\
&= \frac{\partial F}{\partial y'}\eta\Bigg|_a^b + \int_a^b \eta\left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}\right)\mathrm{d}x \\
&= \int_a^b \eta\left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\mathrm{d}x = 0
\end{aligned}$$

Then we get the **Euler's Equation**

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = 0$$

3.2 Techniques of the Calculus of Variations

$$J[y(x)] = \int_a^b F(x, y, y')\mathrm{d}x$$

$$\begin{aligned}
\delta J[y] &= J[y + \delta y] - J[y] \\
&= \int_a^b \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right)\mathrm{d}x \\
&= \frac{\partial F}{\partial y'}\delta y\Bigg|_{x_1}^{x_2} + \int_a^b \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}\right)\delta y\mathrm{d}x = 0 \\
&= \int_a^b \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right)\delta y\mathrm{d}x = 0
\end{aligned}$$

Then we get the **Euler's Equation**

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = 0$$

4 Hamiltonian Dynamics

4.1 Hamilton's Equations (Canonical Equations)

Now let's derive **Hamiltonian** From Lagrangian. The total time derivative of L is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t}$$

But from Lagrange's equations,

$$\frac{\partial L}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$$

And the total time derivative of L can be written as

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_j \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) + \frac{\partial L}{\partial t}$$

It therefore follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L\right) + \frac{\partial L}{\partial t} = \frac{\mathrm{d}H}{\mathrm{d}t} + \frac{\partial L}{\partial t} = 0$$

or

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial L}{\partial t}$$

We define that

$$H = \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L = \sum_j p_j\dot{q}_j - L$$

The function H is called the Hamiltonian of the system.

4.1.1 Derive Hamilton's Equations by Differential Way

$$H = H(p_i, q_i, t) \;\Rightarrow\; \mathrm{d}H = \sum_i \left(\frac{\partial H}{\partial p_i}\mathrm{d}p_i + \frac{\partial H}{\partial q_i}\mathrm{d}q_i\right) + \frac{\partial H}{\partial t}\mathrm{d}t$$

$$\begin{aligned} H = \sum_i p_i\dot{q}_i - L\left(q_i, \dot{q}_i, t\right) \;\Rightarrow\; \mathrm{d}H &= \sum_i \left(p_i\mathrm{d}\dot{q}_i + \dot{q}_i\mathrm{d}p_i - \frac{\partial L}{q_i}\mathrm{d}q_i - \frac{\partial L}{\dot{q}_i}\mathrm{d}\dot{q}_i\right) - \frac{\partial L}{\partial t}\mathrm{d}t \\ &= \sum_i \left(\dot{q}_i\mathrm{d}p_i - \dot{p}_i\mathrm{d}q_i\right) - \frac{\partial L}{\partial t}\mathrm{d}t \end{aligned}$$

Then we get the **Hamilton's Equation**

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q} \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

4.1.2 Derive Hamilton's Equations by Legendre Transform

What is a Legendre Transformation?

$$\frac{\partial f}{\partial x_i} = u_i, \quad \frac{\partial f}{\partial y_i} = v_i \quad (i = 1, 2, \ldots, n) \tag{4.1}$$

$$\mathrm{d}f = \sum_i \left(u_i\mathrm{d}x_i + v_i\mathrm{d}y_i\right) + \frac{\partial f}{\partial t}\mathrm{d}t \tag{4.2}$$

We define that

$$g \equiv \sum_i u_i x_i - f \tag{4.3}$$

Then we do the Legendre Transformations:

$$\begin{aligned} \mathrm{d}g &= \sum_i \left(u_i\mathrm{d}x_i + x_i\mathrm{d}u_i\right) - \mathrm{d}f \\ &= \sum_i \left(u_i\mathrm{d}x_i + x_i\mathrm{d}u_i - u_i\mathrm{d}x_i - v_i\mathrm{d}y_i\right) - \frac{\partial f}{\partial t}\mathrm{d}t \\ &= \sum_i \left(x_i\mathrm{d}u_i - v_i\mathrm{d}y_i\right) - \frac{\partial f}{\partial t}\mathrm{d}t \end{aligned} \tag{4.4}$$

$$\mathrm{d}g = \sum_i \left(\frac{\partial g}{\partial u_i}\mathrm{d}u_i + \frac{\partial g}{\partial y_i}\mathrm{d}y_i\right) + \frac{\partial g}{\partial t}\mathrm{d}t$$

According to the Differential Laws, we have

$$\frac{\partial g}{\partial u_i} = x_i \quad \frac{\partial g}{\partial y_i} = -v_i \quad \frac{\partial g}{\partial t} = -\frac{\partial f}{\partial t}$$

Then we can do this

$$H \to g,\; L \to f,\; \dot{q} \to x,\; q \to y,\; p \to u,\; \dot{p} \to v$$

Then we have

$$H = \sum_i p_i \dot{q}_i - L\left(q_i, \dot{q}_i, t\right)$$

$$\frac{\partial L}{\partial \dot{q}_i} = p_i \quad \frac{\partial L}{\partial q} = \dot{p}_i$$

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q} \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Hamiltonian as a Legendre Transform of Lagrangian

4.1.3 Derive Hamilton's Equations From Hamilton's Principle

Hamilton's Principle

$$\delta I \equiv \delta \int_{t_1}^{t_2} L \mathrm{d}t = 0$$

$$L(q, \dot{q}, t) = p\dot{q} - H(q, p, t)$$

$$\delta I = \int_{t_1}^{t_2} \delta(p\dot{q} - H(q,p))\mathrm{d}t = \int_{t_1}^{t_2} \left(p\delta\dot{q} + \dot{q}\delta p - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p\right) \mathrm{d}t = 0$$

The first part can be written like this

$$\int_{t_1}^{t_2} (p\delta\dot{q})\mathrm{d}t = \int_{t_1}^{t_2} \left(p\frac{d}{dt}\delta q\right) \mathrm{d}t = p\delta q\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}\delta q \mathrm{d}t = -\int_{t_1}^{t_2} \dot{p}\delta q \mathrm{d}t$$

Then we have

$$\delta I = \int_{t_1}^{t_2} \left[-\left(\dot{p} + \frac{\partial H}{\partial q}\right)\delta q + \left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p\right] \mathrm{d}t = 0$$

Since the sets of virtual displacement δq and δp are independent, the only way for the equation above to hold is that

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

4.2 Phase Space

$$\dot{q} = \frac{\partial H(p,q)}{\partial p} \qquad \dot{p} = -\frac{\partial H(q,p)}{\partial q} \qquad \frac{\mathrm{d}p}{\mathrm{d}q} = f(q,p)$$

4.2.1 Liouville's Theorem

Denote the density of particles in phase space: $D = D(q,p,t)$

$$\frac{\mathrm{d}D}{\mathrm{d}t} = 0$$

5 The Poisson Bracket

5.1 The Poisson Bracket

$$[\phi, \psi]_{q,p} = \sum_k \left(\frac{\partial \phi}{\partial q_k}\frac{\partial \psi}{\partial p_k} - \frac{\partial \phi}{\partial p_k}\frac{\partial \psi}{\partial q_k}\right)$$

- $[\phi, \psi] = -[\psi, \phi]$
- $[a\phi + b\psi, \theta] = a[\phi, \theta] + b[\psi, \theta]$
- $[\phi, c] = 0$
- $[\phi, \phi] = 0$

- $[q_l, q_s] = 0 \qquad [p_k, p_s] = 0 \qquad [q_k, p_s] = \delta_{ks} = \begin{cases} 1, k = s \\ 0, k \neq s \end{cases}$

  $[q_k, \phi] = \frac{\partial \phi}{\partial p_k} \qquad [p_k, \phi] = -\frac{\partial \phi}{\partial q_k}$
- $[\theta, \psi\phi] = \psi[\theta, \phi] + [\theta, \psi]\phi \qquad [\psi\phi, \theta] = \psi[\phi, \theta] + [\psi, \theta]\phi$
- $[-\phi, \psi] = [\phi, -\psi] = -[\phi, \psi]$
- $\frac{\partial}{\partial t}[\phi, \psi] = \left[\frac{\partial \phi}{\partial t}, \psi\right] + \left[\phi, \frac{\partial \psi}{\partial t}\right] \qquad \frac{\mathrm{d}}{\mathrm{d}t}[\phi, \psi] = \left[\frac{\mathrm{d}\phi}{\mathrm{d}t}, \psi\right] + \left[\phi, \frac{\mathrm{d}\psi}{\mathrm{d}t}\right]$
- Jacobi's identity

$$[\theta, [\phi, \psi]] + [\phi, [\psi, \theta]] + [\psi, [\theta, \phi]] = 0$$

5.2 Fundamental Poisson Bracket

$$[q_j, q_k] = \frac{\partial q_j}{\partial q_i}\frac{\partial q_k}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial q_k}{\partial q_i} = 0 \qquad [p_j, p_k] = 0$$
$$[q_j, p_k] = \frac{\partial q_j}{\partial q_i}\frac{\partial p_k}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial p_k}{\partial q_i} = \delta_{jk} \qquad [p_j, q_k] = -\delta_{jk}$$

5.3 The Equations of Motion

The total time derivative of a function $u(q, p, t)$

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= \sum_k \left(\frac{\partial u}{\partial q_k}\dot{q}_k + \frac{\partial u}{\partial p_k}\dot{p}_k\right) + \frac{\partial u}{\partial t} \\ &= \sum_k \left(\frac{\partial u}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial H}{\partial q_k}\right) + \frac{\partial u}{\partial t} \\ &= [u, H] + \frac{\partial u}{\partial t} \end{aligned}$$

5.4 Poisson Equations

$$\dot{q}_i = [q_i, H] \qquad \dot{p}_i = [p_i, H]$$

5.5 Poisson's Theorem

$$\phi(q, p) = c_1 \quad \psi(q, p) = c_2 \;\Rightarrow\; [\phi, \psi] = c_3$$

Example: $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

$$L_x = yp_z - zp_y \qquad L_y = zp_x - xp_z \qquad L_z = xp_y - yp_x$$
$$L_x = [L_y, L_z] \qquad L_y = [L_z, L_x] \qquad L_z = [L_x, L_y]$$

if L_x, L_y are constants of motion, then L_z is also one.

6 Canonical Transformation

To find the way to optimize the choice of coordinates for maximizing the number of cyclic variables, we suppose

$$Q_i = Q_i(q, p, t) \qquad P_i = P_i(q, p, t)$$

We require that there exits some function $K(Q, P, t)$ such that

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

We know that the canonical equations resulted from the condition

$$\delta \int_{t_1}^{t_2} L\mathrm{d}t = \delta \int_{t_1}^{t_2} \left[\sum_i p_i \dot{q}_i - H(q, p, t)\right] \mathrm{d}t = 0$$

which we can similarly write

$$\delta\int_{t_1}^{t_2} L'\mathrm{d}t = \delta\int_{t_1}^{t_2}\left[\sum_i P_i\dot{Q}_i - K(q,p,t)\right]\mathrm{d}t = 0$$

For the same system, we have

$$L = L' + \frac{\mathrm{d}F}{\mathrm{d}t}$$

F is called the generating function(generator) of the transformation, and it can be any function of p_i, q_i, P_i, Q_i and t.

$$p_i\dot{q}_i - H(q,p,t) = P_i\dot{Q}_i - K(q,p,t) + \frac{\mathrm{d}F}{\mathrm{d}t}$$

Multiplying by the time differential:

$$\mathrm{d}F = \sum_{i=1}^{s}(p_i\mathrm{d}q - P_i\mathrm{d}Q_i) + (K-H)\mathrm{d}t$$

So we have the **Canonical Transformations**:

$$p_i = \frac{\partial F}{\partial q_i} \quad P_i = -\frac{\partial F}{\partial Q_i} \quad K - H = \frac{\partial F}{\partial t}$$

The standard for Canonical Transformations

$$Q = Q(q,p) \qquad P = P(q,p)$$

$$Q = \frac{\partial K}{\partial P} \qquad P = -\frac{\partial K}{\partial Q}$$

$$[Q,P]_{q,p} = 1$$

6.1 Four Basic Generators

$$F = F_1(q,Q,t) \quad p_i = \frac{\partial F}{\partial q_i} \quad P_i = -\frac{\partial F}{\partial Q_i}$$

We have three additional choices by Legendre Transformation $g(y,u) = f(y,x) - ux$

$$F = F_2(q,P,t) - \sum_{i=1}^{s} Q_iP_i$$

$$F = F_3(p,Q,t) + \sum_{i=1}^{s} q_ip_i$$

$$F = F_2(p,P,t) + \sum_{i=1}^{s}(q_ip_i - Q_iP_i)$$

So we have four basic types of generating functions:

$$F_1(q,Q,t) \quad F_2(q,P,t) \quad F_3(p,Q,t) \quad F_4(p,P,t)$$

Generator	Derivatives	Trivial Case
$F_1(q,Q,t)$	$p_i = \frac{\partial F_1}{\partial q_i} \quad P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = \sum_{i=1}^{s} q_iQ_i \quad Q_i = p_i \quad P_i = -q_i$
$F_2(q,P,t) - \sum_{i=1}^{s} Q_iP_i$	$p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = \sum_{i=1}^{s} q_iP_i \quad Q_i = q_i \quad P_i = p_i$
$F_3(p,Q,t) + \sum_{i=1}^{s} q_ip_i$	$q_i = -\frac{\partial F_3}{\partial p_i} \quad P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = \sum_{i=1}^{s} p_iQ_i \quad Q_i = -q_i \quad P_i = -p_i$
$F_4(p,P,t) + \sum_{i=1}^{s}(q_ip_i - Q_iP_i)$	$q_i = -\frac{\partial F_4}{\partial p_i} \quad Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = \sum_{i=1}^{s} p_iP_i \quad Q_i = p_i \quad P_i = -q_i$

6.2 Hamilton-Jacobi Equation

$$H(q, \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = L$$

$$S = \int L\mathrm{d}t$$

$$S = -Et + W(q, P)$$

$$H(q, \frac{\partial W}{\partial q}) = E$$

Part III
Applications

7 Central Force Motion

7.1 Tow-body Problem

$$\text{Reduced Mass}: \ \mu = \frac{m_1 m_2}{m_1 + m_2} \qquad M = m_1 + m_2$$

$$\boldsymbol{r} = \boldsymbol{r_1} - \boldsymbol{r_2} \qquad \boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2}$$

Lagrangian:

$$L = \frac{1}{2} M \dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu \dot{\boldsymbol{r}}^2 - V(\boldsymbol{r}) = \frac{\boldsymbol{P}^2}{2M} + \frac{1}{2}\mu \dot{\boldsymbol{r}}^2 - V(\boldsymbol{r})$$

As we know, $\boldsymbol{P} = cste$, so the re-gauged Lagrangian

$$L' = \frac{1}{2}\mu \dot{\boldsymbol{r}}^2 - V(\boldsymbol{r})$$

From the Noether's Theorem: A central force produces no torque about the center, and the space is isotropic, so the angular momentum about the center is conserved. That is

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = cste$$

Since in this case $\boldsymbol{L}$ is fixed, it follows that the motion is at all time confine to the aforementioned plane.

$$L = \frac{1}{2}\mu \dot{\boldsymbol{r}}^2 - V(\boldsymbol{r}) = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

The momentum p_θ is a **first integral of motion** and is seen to equal the magnitude of the angular momentum vector.

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = cste = l$$

7.2 The Equations of Motion

$$\begin{cases} \mu(\ddot{r} - r\dot{\theta}^2) = F(r) & \text{(1) (Radial equation)} \\ \mu r^2 \dot{\theta} = l & \text{(2) (Lateral equation)} \\ \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V = E & \text{(3) (Conservation of mechanical energy)} \end{cases}$$

7.2.1 The First Way: Orbit Equation

WIth the equations (2) and (3), we get

$$\boxed{\theta = \pm \int \frac{(l/r^2)\mathrm{d}r}{\sqrt{2\mu\left[E - V(r) - \frac{l^2}{2\mu r^2}\right]}} + cste}$$

7.2.2 The Second Way: J.P.Binet Equation

with the equations (1) and (2), we get

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2} F(r)$$

We now modify the equation by making the following change of variable

$$u = \frac{1}{r}$$

then we have the **J.P.Binet Equation**

$$\boxed{l^2 u^2 \left(\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u\right) = -\mu F(r)}$$

7.3 The Characteristics of Orbits

- Total energy

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + V(r) = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r)$$

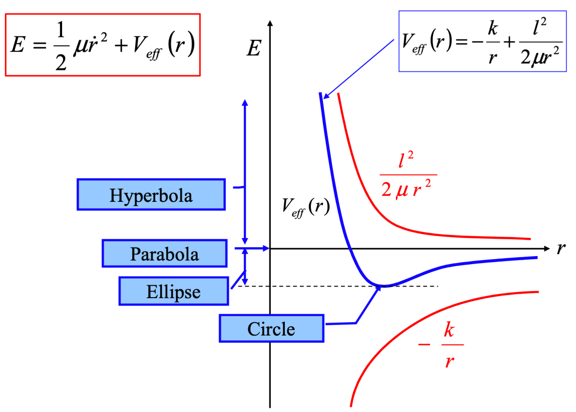
- Rotation potential (centrifugal potential energy)

$$V_c = \frac{1}{2}\frac{l^2}{\mu r^2}$$

- Effective potential

$$V_{\text{eff}} = V(r) + \frac{l^2}{2\mu r^2} = -\int F(r)\mathrm{d}r + \frac{l^2}{2\mu r^2}$$

7.4 Planetary Motion - Kepler's Problem



$\varepsilon > 1$	$E > 0$	Hyperbola
$\varepsilon = 1$	$E = 0$	Pararbola
$0 < \varepsilon < 1$	$V_{min} < E < 0$	Ellipse
$\varepsilon = 0$	$E = V_{min}$	Circle

Orbit Equation:

$$\frac{1}{r} = C(1 + \varepsilon\cos\theta) \qquad C = \frac{\mu k}{l^2} \qquad \varepsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}}$$

$$\frac{1}{r} = \frac{\mu k}{l^2}\left(1 + \sqrt{1 + \frac{2El^2}{\mu k^2}}\cos\theta\right)$$

Orbit parameter:

$$a = -\frac{k}{2E} \qquad b = \sqrt{-\frac{l^2}{2\mu E}}$$

Period of rotation:

$$\tau = 2\pi\sqrt{\frac{\mu}{k}}a^{\frac{3}{2}}$$

7.4.1 Stable Circular Orbits

1. V_{eff}

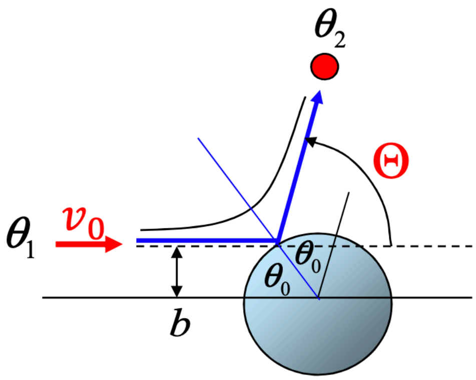
$$\left.\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r}\right|_{r=r_0} = 0 \qquad \left.\frac{\mathrm{d}^2V_{\text{eff}}}{\mathrm{d}r^2}\right|_{r=r_0} > 0$$

2. linearization ($r = r_0 + x$)

$$\ddot{x} + \frac{1}{m}\left.\frac{\mathrm{d}^2V_{\text{eff}}(r)}{\mathrm{d}r^2}\right|_{r=r_0} x = 0$$

$$\omega_r^2 = \frac{1}{m}\frac{\mathrm{d}^2V_{\text{eff}}(r)}{\mathrm{d}r^2} > 0$$

7.5 Scattering



- Force field

$$F(r) = \frac{k}{r^2} \qquad k = \frac{q_1 q_2}{4\pi\varepsilon_0}$$

- Orbit Equation:

$$\frac{1}{r} = C(-1 + \varepsilon\cos\theta) \qquad C = \frac{\mu k}{l^2} \qquad \varepsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}}$$

$$\frac{1}{r} = \frac{\mu k}{l^2}\left(-1 + \sqrt{1 + \frac{2El^2}{\mu k^2}}\cos\theta\right)$$

- The angle of scattering:

$$\Theta = \pi - (\theta_2 - \theta_1) \qquad\qquad \cot\frac{\Theta}{2} = \frac{2Eb}{k}$$

- Differential Cross Section:

$$\sigma(\Theta) = \frac{I'}{I} \qquad \text{where } I' = \frac{\mathrm{d}N}{\mathrm{d}\Omega}$$

$$\sigma(\Theta) = -\frac{b}{\sin\Theta}\frac{\mathrm{d}b}{\mathrm{d}\Theta}$$

$$\sigma = \int \sigma(\Theta)\mathrm{d}\Omega = \int_0^\pi 2\pi\sigma(\Theta)\sin\Theta\mathrm{d}\Theta$$

8 Dynamics of Rigid Bodies

8.1 The Inertia Tensor and The Kinetic Energy

$$I = \int r^2 \mathrm{d}m$$

Inertia tensor $\mathbf{I}$

$$\mathbf{I} = \begin{bmatrix} \int(y^2+z^2)\mathrm{d}m & -\int xy\mathrm{d}m & -\int xz\mathrm{d}m \\ -\int xy\mathrm{d}m & \int(z^2+x^2)\mathrm{d}m & -\int yz\mathrm{d}m \\ -\int xz\mathrm{d}m & -\int yz\mathrm{d}m & \int(x^2+y^2)\mathrm{d}m \end{bmatrix}$$

The principal Axes of inertia

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$$\mathbf{L} = \mathbf{I}\omega$$

$$T = \frac{1}{2}\boldsymbol{\omega}^T\mathbf{I}\boldsymbol{\omega}$$

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

8.2 Euler Angles

$$\begin{array}{lll}
(x', y', z') & & x \\
\downarrow & \text{Rotate by } \phi \text{ around } z' \text{ axis} & \\
(\xi, \eta, \zeta) & & \mathbf{D}x \\
\downarrow & \text{Rotate by } \theta \text{ around } \xi \text{ axis} & \\
(\xi', \eta', \zeta') & & \mathbf{CD}x \\
\downarrow & \text{Rotate by } \psi \text{ around } \zeta' \text{ axis} & \\
(x, y, z) & & \mathbf{A}x = \mathbf{BCD}x
\end{array}$$

$$\mathbf{D} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

In Inertia principal axes system

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \sin\psi\sin\theta \\ \cos\psi\sin\theta \\ \cos\theta \end{bmatrix} \dot{\phi} + \begin{bmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{bmatrix} \dot{\theta} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\psi} = \begin{bmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\phi} + \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} \dot{\theta} + \begin{bmatrix} \sin\theta\sin\phi \\ -\sin\theta\cos\phi \\ \cos\theta \end{bmatrix} \dot{\psi} = \begin{bmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\ \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\ \dot{\phi} + \dot{\psi}\cos\theta \end{bmatrix}$$

$$\begin{aligned} T &= \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) \\ &= \frac{1}{2}I_1\left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)^2 + \frac{1}{2}I_2\left(\dot{\phi}\sin\theta\cos\psi - \dot{\phi}\sin\psi\right)^2 + \frac{1}{2}I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 \end{aligned}$$

- In Inertia principal axes system

$$L = T - V = \frac{1}{2}I_1\left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)^2 + \frac{1}{2}I_2\left(\dot{\phi}\sin\theta\cos\psi - \dot{\phi}\sin\psi\right)^2 + \frac{1}{2}I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 - V$$

- If $I_1 = I_2$

$$L = \frac{1}{2}I_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - V$$

- if $I_1 = I_2 = I_3 = I$

$$L = \frac{1}{2}I\left(\dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta\right) - V$$

8.3 Euler's Equations

$$\dot{\mathbf{L}} = \mathbf{M} + \mathbf{L} \times \omega$$

In Inertia principal axes system

$$\begin{cases} I_1\dot{\omega}_1 = M_1 + (I_2 - I_3)\,\omega_2\omega_3 \\ I_2\dot{\omega}_2 = M_2 + (I_3 - I_1)\,\omega_3\omega_1 \\ I_3\dot{\omega}_3 = M_3 + (I_1 - I_2)\,\omega_1\omega_2 \end{cases}$$

8.4 Lagrangian Method for Rigid Dynamics

- Asymmetrical top: $I_1 \neq I_2 \neq I_3$
- Symmetrical top: $I_1 = I_2 \neq I_3$
- Spherical top: $I_1 = I_2 = I_3$
- Rotator: $I_1 = I_2 \neq 0 \quad I_3 = 0$

8.4.1 Rotationanl Kinetic Energy of a Symmetric Top

The rotational kinetic energy for a symmetric top can be written as

$$T_{\text{rot}} = \frac{1}{2}I_1\left(\omega_1^2+\omega_2^2\right)+\frac{1}{2}I_3\omega_3^2 = \frac{1}{2}I_1\left(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta\right)+\frac{1}{2}I_3\left(\dot{\psi}+\dot{\phi}\cos\theta\right)^2$$

$$L(\theta,\dot{\theta},\dot{\phi},\dot{\psi}) = T_{\text{rot}}$$

SInce ϕ and ψ are ignorable coordinates, there canonical angular momenta

$$\begin{aligned}
p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = I_1\dot{\phi}\sin^2\theta + I_3\left(\dot{\psi}+\dot{\phi}\right)\cos\theta = cste \\
p_\psi &= \frac{\partial L}{\partial \dot{\psi}} = I_3\omega_3 = cste \\
p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = I_1\dot{\theta}
\end{aligned}$$

p_ϕ and ψ are constants of the motion. By inverting these relations, we obtain

$$\dot{\phi} = \frac{p_\theta - p_\psi\cos\theta}{I_1\sin^2\theta} \qquad \dot{\psi} = \omega_3 - \frac{(p_\phi - p_\psi\cos\theta)\cos\theta}{I_1\sin^2\theta}$$

8.4.2 Symmetric Top with One Fixed Point

We now consider the case of a spinning symmetric top of mass M and principal moments of intertia ($I_1 = I_2 \neq I_3$) with one fixed point O moving in a gravitational field with constant acceleration g.

$$L = \frac{1}{2}I_1\left(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta\right)+\frac{1}{2}I_3\left(\dot{\psi}+\dot{\phi}\cos\theta\right)^2 - Mgh\cos\theta$$

$$V_{\text{eff}}(\theta) = \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta$$

8.4.3 Stability of the Sleeping Top

Let's consider the case where a symmetric top with one fixed point is launched with initial conditions $\theta_0 \neq 0$ and $\dot{\theta} = \dot{\phi} = 0$, with $\dot{\psi} \neq 0$. In this case, the invariant canonical momenta are

$$p_\psi = I_3\dot{\psi}_0 \qquad p_\phi = p_\psi\cos\theta_0$$

9 Small Oscillations

9.1 frequency of oscillation

$$T = \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}}$$

$$V = \frac{1}{2}\sum_{j,k}\left(\frac{\partial^2 V}{\partial q_j\partial q_k}\right)q_jq_k = \frac{1}{2}\sum_{j,k} v_{jk}q_jq_k = \frac{1}{2}\mathbf{q}^T\mathbf{V}\mathbf{q}$$

Thereinto

$$\mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \qquad \dot{\mathbf{q}} = \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \cdots & M_{nn} \end{pmatrix} \qquad \mathbf{V} = \begin{pmatrix} V_{11} & \cdots & V_{1n} \\ \vdots & \ddots & \vdots \\ V_{n1} & \cdots & V_{nn} \end{pmatrix}$$

We than get the Lagrangian

$$L = T - V = \frac{1}{2}\sum_{j,k}(m_{jk}\dot{q}_j\dot{q}_k - v_{jk}q_jq_k) = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} - \frac{1}{2}\mathbf{q}^T\mathbf{V}\mathbf{q}$$

Using the Lagrangian Equation

$$\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

We get

$$\sum_j (m_{jk}\ddot{q}_j + v_{jk}q_j) = 0$$

And they can be written in a matrix form:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{V}\mathbf{q} = 0$$

We suppose

$$\mathbf{q} = \mathbf{A}e^{i\omega t} \qquad \text{and} \qquad \mathbf{A} = (A_1, \cdots, A_n)^T$$

Then we get

$$(\mathbf{V} - \omega^2\mathbf{M})\mathbf{A} = 0$$

In order to get a non-trivial solution to this equation, the determinant of the quantity in parentheses must vanish

$$\det(\mathbf{V} - \omega^2\mathbf{M}) = 0$$

This determinant is called the **characteristic or secular equation** and is an equation of degree n in ω^2. The corresponding n roots ω_r^2 are the **characteristic frequencies** or **eigenfrequencies**. The eigenvector $\mathbf{A}_r = (A_{1r}, \cdots, A_{nr})^T$ associated with a given root ω_r. We can write the generalized coordinate q_j as a linear combination of the solutions for each root

$$\mathbf{q} = \sum_r \mathbf{A}_r c_r e^{i(\omega_r t - \delta_r)} \qquad \text{or} \qquad \mathbf{q} = \sum_r \mathbf{A}_r c_r \cos(\omega_r t - \delta_r)$$

9.2 Normal Coordinates

$$X_r = c_r \cos(\omega_r - \delta_r)$$

$$L = \sum_r \frac{1}{2} m_r (\dot{X}_r^2 - \omega_r^2 X_r)$$

$$\mathbf{q} = \sum_r \mathbf{A}_r X_r$$

or

$$\mathbf{q} = \mathbf{A}'\mathbf{X}$$

$$\mathbf{A}' = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$