Revision for 2024 MMP Exam

Chen Huang*
Department of Physics, Imperial College London

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(*) means very important.

1 Vector Spaces and Tensors

(1) **Summation convention**

$$C_{ij} = \sum_k A_{ik}B_{kj} = A_{ik}B_{kj} \tag{1}$$

where i, j are the free indices; k is the dummy index.

Note: in any one term of an expression, indices may appear 0, 1, 2 times.

(2) **Levi-Civita symbol**

$$\varepsilon_{ijk} = \begin{cases} 1, & (i,j,k) = (1,2,3), (2,3,1), (3,1,2) \\ -1, & (i,j,k) = (3,2,1), (2,1,3), (1,3,2) \\ 0, & \text{Otherwise} \end{cases} \tag{2}$$

Two important relations:

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} \quad \Leftrightarrow \quad c_i = \varepsilon_{ijk}a_j b_k \tag{3}$$

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{4}$$

(3) **Tensor calculus**

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = (\partial_i, \partial_j, \partial_k) \tag{5}$$

- Gradient

$$(\boldsymbol{\nabla}\phi)_i = \partial_i \phi \tag{6}$$

- Divergence

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \partial_i F_i \tag{7}$$

*chen.huang23@imperial.ac.uk

- Curl
$$(\boldsymbol{\nabla} \times \boldsymbol{F})_i = \varepsilon_{ijk}\partial_j F_k \tag{8}$$

(4) **Transforms under rotations (*)**
The rotation matrix $\mathbf{L}$ is orthogonal, with

$$L_{ij}L_{ik} = L_{ji}L_{ki} = \delta_{jk} \tag{9}$$

- A scalar $\phi(x)$
$$\phi(x) \to \phi'(x') = \phi(x) \tag{10}$$
- A vector $v_i(x)$
$$v_i(x) \to v_i'(x') = L_{ij}v_j(x) \tag{11}$$
- A rank 2 tensor $T_{ij}(x)$
$$T_{ij}(x) \to T_{ij}'(x') = L_{il}L_{jm}T_{lm}(x) \tag{12}$$

2 Green Functions

(1) **Wronskian**
Consider homogeneous second order differential equations

$$y'' + p(x)y' + q(x)y = 0 \tag{13}$$

and $y_1(x)$ and $y_2(x)$ are linearly independent solutions with non-vanishing Wronskian

$$\boxed{W(x) = y_1y_2' - y_1'y_2} \tag{14}$$

Note: $W \neq 0 \;\Leftrightarrow\; y_1$ and y_2 are independent.
Here are two important consequences

$$y_2(x) = y_1(x)\int^x \frac{W(\tilde{x})}{y_1^2(\tilde{x})}\mathrm{d}\tilde{x} \tag{15}$$

$$W(x) = \pm c \exp\left[-\int^x p(\tilde{x})\mathrm{d}\tilde{x}\right] \tag{16}$$

(2) **Inhomogeneous**
$$y'' + p(x)y' + q(x)y = f(x) \tag{17}$$

Then $y(x) = ay_1(x) + by_2(x) + y_0(x)$ is a solution of the inhomogeneous ODE. The particular integral

$$y_0 = u(x)y_1(x) + v(x)y_2(x) \tag{18}$$

subject to the constraint

$$\boxed{u'y_1 + v'y_2 = 0} \tag{19}$$

and the ODE is simplified as

$$\boxed{u'y_1' + v'y_2' = f} \tag{20}$$

So the solutions are

$$u'(x) = -\frac{y_2(x)f(x)}{W(x)}, \qquad v'(x) = \frac{y_1(x)f(x)}{W(x)} \tag{21}$$

(3) **Green function** (*)

$$\boxed{\mathcal{L}_x G(x,\tilde{x}) = \delta(x-\tilde{x})} \tag{22}$$

$G(x,\tilde{x})$ is continuous at $x=\tilde{x}$

$$[G(x,\tilde{x})]_{x\to\tilde{x}_-}^{x\to\tilde{x}_+} = 0 \tag{23}$$

$\frac{\partial}{\partial x}G(x,\tilde{x})$ has a unit discontinuity at $x=\tilde{x}$

$$\left[\frac{\partial G(x,\tilde{x})}{\partial x}\right]_{x\to\tilde{x}_-}^{x\to\tilde{x}_+} = 1 \tag{24}$$

(4) **Boundary conditions**
The particular integral of the OED

$$y_0 = \int_\alpha^\beta \mathrm{d}\tilde{x} G(x,\tilde{x}) f(\tilde{x}) \tag{25}$$

a) **Homogeneous initial conditions** $y(\alpha) = y'(\alpha) = 0$.

- For $x < \tilde{x}$, $G(x,\tilde{x}) = 0$ is a solution of the homogeneous equation that satisfies the boundary conditions $G(\alpha,\tilde{x}) = G'(\alpha,\tilde{x}) = 0$.
- For $x > \tilde{x}$, $G(x,\tilde{x}) = A(\tilde{x})y_1(x) + B(\tilde{x})y_2(x)$.
  i) G is continuous at $x = \tilde{x}$

$$A(\tilde{x})y_1(\tilde{x}) + B(\tilde{x})y_2(\tilde{x}) = 0 \tag{26}$$

  ii) G' has a unit discontinuity at $x=\tilde{x}$

$$A(\tilde{x})y_1'(\tilde{x}) + B(\tilde{x})y_2'(\tilde{x}) = 1 \tag{27}$$

  The solution is

$$A(\tilde{x}) = -\frac{y_2(\tilde{x})}{W(\tilde{x})}, \qquad B(\tilde{x}) = \frac{y_1(\tilde{x})}{W(\tilde{x})} \tag{28}$$

b) **Homogeneous two-point boundary Conditions** $y(\alpha) = y(\beta) = 0$.

$$G(x,\tilde{x}) = \begin{cases} A(\tilde{x})y_1(x) + B(\tilde{x})y_2(x), & x < \tilde{x} \\ C(\tilde{x})y_1(x) + D(\tilde{x})y_2(x), & x > \tilde{x} \end{cases} \tag{29}$$

- Boundary conditions: $G(\alpha, \tilde{x}) = G(\beta, \tilde{x}) = 0$

$$A(\tilde{x})y_1(\alpha) + B(\tilde{x})y_2(\alpha) = B(\tilde{x})y_2(\alpha) = 0 \quad \Rightarrow \quad B(\tilde{x}) = 0 \tag{30}$$

$$C(\tilde{x})y_1(\beta) + D(\tilde{x})y_2(\beta) = C(\tilde{x})y_1(\beta) = 0 \quad \Rightarrow \quad C(\tilde{x}) = 0 \tag{31}$$

so we have

$$G(x, \tilde{x}) = \begin{cases} A(\tilde{x})y_1(x), & \alpha < x < \tilde{x} \\ D(\tilde{x})y_2(x), & \tilde{x} < x < \beta \end{cases} \tag{32}$$

- Continuity of G and unit discontinuity of G' at $x = \tilde{x}$

$$A(\tilde{x})y_1(\tilde{x}) - D(\tilde{x})y_2(\tilde{x}) = 0 \tag{33}$$

$$A(\tilde{x})y_1'(\tilde{x}) - D(\tilde{x})y_2'(\tilde{x}) = 1 \tag{34}$$

so we have

$$A(\tilde{x}) = \frac{y_2(\tilde{x})}{W(\tilde{x})}, \qquad D(\tilde{x}) = \frac{y_1(\tilde{x})}{W(\tilde{x})} \tag{35}$$

3 Sturm-Liouville Theory

(1) **Self-adjoint form** (*)

$$\boxed{\mathcal{L} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + \sigma(x)} \tag{36}$$

where $\rho(x) > 0$ and $x \in (a, b)$.
Note: being in self-adjoint form does NOT mean that a differential operator is self-adjoint, that depend on the operator and the specific Hilbert space.

(2) **Self-adjoint** (*)
A second order linear differential operator $\mathcal{L}$ is self-adjoint on Hilbert space $\mathcal{H}$ if

$$\boxed{\langle u, \mathcal{L}v\rangle = \langle v, \mathcal{L}u\rangle^*, \quad \forall u, v \in \mathcal{H}} \tag{37}$$

Consider $\mathcal{L}$ as in self-adjoint form,

$$\begin{aligned}
\langle u, \mathcal{L}v\rangle &= \int_a^b u^* \left[-(\rho v')' + \sigma v\right] \mathrm{d}x \\
&= -\left. u^*\rho v'\right|_a^b + \int_a^b \left(u^{*\prime}\rho v' + u^*\sigma v\right) \mathrm{d}x \\
&= -\left. u^*\rho v'\right|_a^b + \left. u^{*\prime}\rho v\right|_a^b + \int_a^b \left(-(u^{*\prime}\rho)'v + u^*\sigma v\right) \mathrm{d}x \\
&= \left.(-u^*\rho v' + u^{*\prime}\rho v)\right|_a^b + \int_a^b \left(-(u^{*\prime}\rho)' + u^*\sigma\right) v\mathrm{d}x \\
&= \left.(-u^*\rho v' + u^{*\prime}\rho v)\right|_a^b + \left[\int_a^b \left(-(u'\rho)' + u\sigma\right) v^*\mathrm{d}x\right]^* \\
&= \left.(-u^*\rho v' + u^{*\prime}\rho v)\right|_a^b + \langle v, \mathcal{L}u\rangle^*
\end{aligned} \tag{38}$$

$\mathcal{L}$ is self-adjoint on $\mathcal{H}$ if

$$\boxed{\rho\left[u^{*\prime}v - u^{*}v'\right]_a^b = 0} \tag{39}$$

(3) **Weight function** (*)
Consider the most generally operator

$$\boxed{\tilde{\mathcal{L}} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(A(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) - B(x)\frac{\mathrm{d}}{\mathrm{d}x} + C(x)} \tag{40}$$

where $x \in (a,b)$. A, B, C are real and $A(x) > 0$. We want to find the weight function, w, such that $\mathcal{L} = w\tilde{\mathcal{L}}$ is in self-adjoint form, *i.e.*

$$-(\rho y')' + \sigma y = w\left[-(Ay')' - By' + Cy\right] \tag{41}$$

gives

$$\frac{w'}{w} = \frac{B}{A}, \qquad \rho = Aw, \qquad \sigma = Cw \tag{42}$$

then we can choose $w(x)$ such that

$$\boxed{w(x) = \exp\left[\int_a^x \frac{B(\tilde{x})}{A(\tilde{x})}\mathrm{d}\tilde{x}\right]} \tag{43}$$

where $w(a) = 1$.

(4) **Eigenfunctions and Eigenvalues** (*)

$$\tilde{\mathcal{L}}y = \lambda y \quad \Rightarrow \quad \boxed{\mathcal{L}y = \lambda\omega y} \tag{44}$$

here, y is an eigenfunction of the self-adjoint operator $\mathcal{L}$ with eigenvalue λ and weight w.

a) The eigenvalues λ are real.

b) The eigenfunctions y with distinct eigenvalues are orthogonal.

Proof: Consider two eigenfunctions, y_i and y_j of $\tilde{\mathcal{L}}$ with eigenvalues λ_i and λ_j respectively. They are also eigenfunctions of $\mathcal{L}$ with eigenvalues λ_i and λ_j and weight w. $\mathcal{L}$ is self-adjoint, then

$$\langle y_i, \mathcal{L}y_j\rangle = \lambda_j\langle y_i, wy_j\rangle = \lambda_j\langle y_i, y_j\rangle_w \tag{45}$$

$$\langle y_i, \mathcal{L}y_j\rangle = \langle y_j, \mathcal{L}y_i\rangle^* = \lambda_i^*\langle y_j, wy_i\rangle^* = \lambda_i^*\langle y_i, y_j\rangle_w \tag{46}$$

Compare the two equations, we have

$$(\lambda_j - \lambda_i^*)\langle y_i, y_j\rangle_w = 0 \tag{47}$$

- For $i = j$, we have $(\lambda_i - \lambda_i^*)\|y_i\|_w^2 = 0 \;\Rightarrow\; \lambda_i = \lambda_i^*$. λ_i is real.

- For $i \neq j$, we have $(\lambda_j - \lambda_i)\langle y_i, y_j\rangle_w = 0 \;\Rightarrow\; \langle y_i, y_j\rangle_w = 0$, *i.e.*, the eigenfunctions are orthogonal with weight $w(x)$.

(5) **Eigenfunction Expansions**
A function $f(x)$ can be written as

$$f(x) = \sum_n f_n y_n(x) \tag{48}$$

where y_n is the eigenfunction of $\mathcal{L}$ with the weight w. f_n is the expansion coefficient and

$$f_n = \langle y_n(x), f(x)\rangle_w = \int_a^b y_n^*(x) w(x) f(x) \mathrm{d}x \tag{49}$$

(6) **Monic polynomial**

$$y_n = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{50}$$

4 Integral Transforms

(1) **Fourier transform**

$$\boxed{\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \mathrm{d}x} \tag{51}$$

Convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x-y) \mathrm{d}y \tag{52}$$

Parseval's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \mathrm{d}k \tag{53}$$

Properties

a) $\mathcal{F}[f_1 * f_2] = \sqrt{2\pi}\tilde{f}_1(k)\tilde{f}_2(k)$
b) $\mathcal{F}[f_1(x)f_2(x)] = \frac{1}{\sqrt{2\pi}}\tilde{f}_1(k) * \tilde{f}_2(k)$
c) $\mathcal{F}[f^{(n)}(x)] = (ik)^n \tilde{f}(k)$
d) $\mathcal{F}[x^n f(x)] = \left(i\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \tilde{f}(k)$
e) $\mathcal{F}[f(x-a)] = \mathrm{e}^{-ia}\tilde{f}(k)$

(2) **Laplace transform**

$$\boxed{\hat{f}(s) = \int_0^{\infty} f(t) \mathrm{e}^{-st} \mathrm{d}t} \tag{54}$$

Convolution

$$(f * g)(t) = \int_0^t f(t') g(t-t') \mathrm{d}t' \tag{55}$$

Properties

a) $\mathcal{L}[f_1 * f_2] = \hat{f}_1(s)\hat{f}_2(s)$

b) $\mathcal{L}[f^{(n)}(t)] = s^n \hat{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$

c) $\mathcal{L}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\hat{f}(s)$

5 Complex Analysis

$$z = x + iy, \qquad f(x,y) = u(x,y) + iv(x,y) \tag{56}$$

(1) **Analytic:** $f(z)$ is analytic in a domain D means the function is differentiable at every point in domain D.

(2) **Cauchy-Riemann equations** (*)

$$f(x,y) \text{ is analytic in a domain } D \quad \Leftrightarrow \quad \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Derivative (*)

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\partial z}{\partial x} = f', \qquad \frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial z}{\partial y} = if' \tag{57}$$

which means

$$i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \quad \Rightarrow \quad i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \tag{58}$$

(3) **Harmonic functions $\nabla^2 f = 0$** (*)

$$f(x,y) \text{ is analytic in domain } D \quad \Leftrightarrow \quad u(x,y) \text{ and } v(x,y) \text{ are harmonic}$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

(4) **Multi-valued functions Example:**

$$f(z) = z^{1/2} \tag{59}$$

a) **Brunch cuts:** Choosing the branch cut to lie on (just below) the negative real axis *i.e.* θ to lie between $-\pi < \theta \leq \pi$.

b) **Branches** of $f(z)$ (*)
We define the principal branch Arg(z) $\in (-\pi, \pi]$. The two branches of $f(z)$ are

$$F_1(z) = |z|^{1/2}\mathrm{e}^{i\theta/2} \tag{60}$$
$$F_2(z) = |z|^{1/2}\mathrm{e}^{i\theta/2 + i\pi} = -|z|^{1/2}\mathrm{e}^{i\theta/2} \tag{61}$$

where $\theta = \mathrm{Arg}(z)$. Each branch has a branch cut along the negative real axis where it is discontinuous.

c) **Riemann sheets:** A Riemann sheet is one separate copy of the complex plane for each branch.

d) **Riemann surface:** The Riemann surface for $f(z)$ is formed by the 2 sheets glued together along the branch cuts. Consider going around the origin on a circle anticlockwise on Riemann sheet number 1: as one reaches and crosses the negative read axis, one passes onto Riemann sheet number 2. As one goes around the origin anticlockwise on Riemann sheet number 2, as one reaches and crosses the negative real axis, one returns to sheet number 1.

e) Why construct riemann surface? The reason for defining the Riemann surface is so that the function $f(z)$ is **single valued** and **continuous** everywhere on the domain of the function.

(5) **Cauchy's theorem** (*) If $f(z)$ is analytic everywhere on and within a closed contour C

$$\oint_C f(z)\mathrm{d}z = 0 \tag{62}$$

Proof (*) From Green's theorem in the plane, P and Q are functions of x and y, and C is a closed contour in the $x-y$ plane, then

$$\oint_C (P\mathrm{d}x + Q\mathrm{d}y) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y \tag{63}$$

so we have

$$\begin{aligned}
\oint_C f(z)\mathrm{d}z &= \oint_C (u+iv)(\mathrm{d}x + i\mathrm{d}y) \\
&= \oint_C [(u+iv)\mathrm{d}x + (-v+iu)\mathrm{d}y] \\
&= \iint_D \left[\frac{\partial}{\partial x}(-v+iu) - \frac{\partial}{\partial y}(u+iv)\right] \mathrm{d}x\mathrm{d}y \\
&= \iint_D \left[\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right] \mathrm{d}x\mathrm{d}y = 0
\end{aligned} \tag{64}$$

(6) **Cauchy's integral theorem**: If $f(z)$ is analytic within and on a closed contour C and z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\mathrm{d}z \tag{65}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z \tag{66}$$

(7) **Residue**: Let f has an isolated singularity at z_0, then the residue of f at z_0 is

$$\mathrm{Res}_f(z_0) = \frac{1}{2\pi i}\oint_{C_{z_0}} f(z)\mathrm{d}z \tag{67}$$

where C_{z_0} is a closed contour *s.t.* z_0 is inside and $f(z)$ is analytic inside except at z_0. If $f(z)$ has a pole of order m at z_0, then

$$f(z) = \frac{g(z)}{(z-z_0)^m} \tag{68}$$

and

$$\mathrm{Res}_f(z_0) = \frac{1}{2\pi i}\oint_C \frac{g(z)}{(z-z_0)^m}\mathrm{d}z = \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}g(z)}{\mathrm{d}z^{m-1}}\bigg|_{z=z_0} \tag{69}$$

(8) **Residue theorem** (*) Let C is a closed contour, $f(z)$ is a function that is analytic on C and inside C except at $z = z_1, \cdots, z_N$. Then

$$\oint_C f(z)\mathrm{d}z = 2\pi i \sum_{k=1}^{N} \mathrm{Res}_f(z_k) \tag{70}$$

(9) **Jordan's lemma** (*)

$$I(R) = \int_{C_R} \mathrm{e}^{i\alpha z} f(z)\mathrm{d}z \tag{71}$$

where $\alpha > 0$ ($\alpha < 0$) and C_R is a semicircle of radius R in the upper (lower) half-plane. Let $M(R)$ be the maximum value of $f(z)$ on C_R. If $M(R) \to 0$ as $R \to \infty$, so dose $I(R)$.

(10) **Inverse Laplace transform** (*)

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\mathrm{e}^{st}\mathrm{d}s = \sum_i \mathrm{Res}(a_i) \tag{72}$$

6 Calculus of Variations

(1) **Euler-Lagrange equation** (*)

$$I[y] = \int_{x_A}^{x_B} f(x, y, y')\mathrm{d}x \tag{73}$$

Varying y slightly

$$\begin{aligned} I[y+\delta y] &= \int_{x_A}^{x_B} f(x, y+\delta y, y'+\delta y')\mathrm{d}x \\ &= \int_{x_A}^{x_B} \left[f(x,y,y') + \delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y'} + \mathcal{O}(\delta y^2) \right] \mathrm{d}x \end{aligned} \tag{74}$$

then

$$\begin{aligned} \delta I[y] &= \int_{x_A}^{x_B} \left[\delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y'} \right] \mathrm{d}x \\ &= \left(\delta y \frac{\partial f}{\partial y'} \right)_{x_A}^{x_B} + \int_{x_A}^{x_B} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'} \right) \right] \delta y \mathrm{d}x \\ &= \int_{x_A}^{x_B} \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'} \right) \right] \delta y \mathrm{d}x = 0 \end{aligned} \tag{75}$$

so

$$\boxed{\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'} \right) = 0} \tag{76}$$

(2) **Beltrami identity** (*)

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial f}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}x} \\ &= \frac{\partial f}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)y' + \frac{\partial f}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}x} \\ &= \frac{\partial f}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}y'\right)\end{aligned} \tag{77}$$

Suppose f has no explicit dependence on x, *i.e.*, $\partial f/\partial x = 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f - \frac{\partial f}{\partial y'}y'\right) = 0 \tag{78}$$

which means

$$\boxed{f - \frac{\partial f}{\partial y'}y' = \text{const}} \tag{79}$$