Chapter 2: Path Integral Formalism of Quantum Mechanics

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1 Definition of Propagator

Time Independent Case

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle \tag{1}$$

H 不含时，波函数可以写成 (t' 是 t 之前的任意一个时间点)

$$|\psi(t)\rangle = \exp\left[-\frac{i}{\hbar}H(t-t')\right]|\psi(t')\rangle \tag{2}$$

将 $\langle\vec{r}|$ 作用在方程两边

$$\langle\vec{r}|\psi(t)\rangle = \langle\vec{r}|\exp\left[-\frac{i}{\hbar}H(t-t')\right]|\psi(t')\rangle \tag{3}$$

单位算符 $\int \mathrm{d}\vec{r}'\,|\vec{r}'\rangle\langle\vec{r}'| = 1$ 作用在方程右边

$$\langle\vec{r}|\psi(t)\rangle = \int \mathrm{d}\vec{r}'\,\langle\vec{r}|\exp\left[-\frac{i}{\hbar}H(t-t')\right]|\vec{r}'\rangle\langle\vec{r}'|\psi(t')\rangle \tag{4}$$

$$\psi(\vec{r},t) = \langle\vec{r}|\psi(t)\rangle \tag{5}$$

$$\psi(\vec{r},t) = \int \mathrm{d}\vec{r}'\,\langle\vec{r}|\exp\left[-\frac{i}{\hbar}H(t-t')\right]|\vec{r}'\rangle\,\psi(\vec{r}',t') = \int \mathrm{d}\vec{r}'\,K(\vec{r},t;\vec{r}',t')\psi(\vec{r}',t') \tag{6}$$

在物理上，$K(\vec{r},t;\vec{r}',t')$ 被称为传播子 (propagator)，在数学上是 Kernel 或 Green's function。定义

$$K(\vec{r},t;\vec{r}',t') = \langle\vec{r}|\exp\left[-\frac{i}{\hbar}H(t-t')\right]|\vec{r}'\rangle \tag{7}$$

当 $t=t'$ 时

$$K(\vec{r},t;\vec{r}',t') = \langle\vec{r}|\vec{r}'\rangle = \delta(\vec{r}-\vec{r}') \tag{8}$$

假定 H 存在一系列本征态和相应的本征值

$$H|n\rangle = E_n|n\rangle \tag{9}$$

$|n\rangle$ 构成完备基，存在单位算符

$$|n\rangle\langle n| = 1 \tag{10}$$

$$\begin{aligned}K(\vec{r},t;\vec{r}',t') &= \langle\vec{r}|\exp\left[-\frac{i}{\hbar}H(t-t')\right]|\vec{r}'\rangle \\ &= \sum_n\sum_{n'}\langle\vec{r}|n\rangle\langle n|\exp\left[-\frac{i}{\hbar}H(t-t')\right]|n'\rangle\langle n'|\vec{r}'\rangle \\ &= \sum_n\sum_{n'}\psi_n(\vec{r})\delta_{n,n'}\psi_{n'}^\dagger(\vec{r}')\exp\left[-\frac{i}{\hbar}E_n(t-t')\right] \\ &= \sum_n\psi_n(\vec{r})\psi_n^\dagger(\vec{r}')e^{-\frac{i}{\hbar}E_n(t-t')} \\ &= \sum_n\left[\psi_n(\vec{r})e^{-\frac{i}{\hbar}E_nt}\right]\left[\psi_n(\vec{r}')e^{-\frac{i}{\hbar}E_nt'}\right]^\dagger \\ &= \sum_n\psi_n(\vec{r},t)\psi_n^\dagger(\vec{r}',t')\end{aligned} \tag{11}$$

Example: The Free Particle

$$H = -\frac{\hbar^2}{2m}\nabla^2 \tag{12}$$

自由粒子的解是平面波

$$\psi_{\vec{p}}(\vec{r},t) = \frac{1}{(2\pi\hbar)^{3/2}}\exp\left[\frac{i}{\hbar}\left(\vec{p}\cdot\vec{r}-\frac{p^2}{2m}t\right)\right] \tag{13}$$

其传播子

$$\begin{aligned}K(\vec{r},t;\vec{r}',t') &= \int\psi_{\vec{p}}(\vec{r},t)\psi_{\vec{p}}^\dagger(\vec{r}',t')\mathrm{d}\vec{p} \\ &= \frac{1}{(2\pi\hbar)^3}\int\mathrm{d}\vec{p}\exp\left[\frac{i\vec{p}}{\hbar}(\vec{r}-\vec{r}')-\frac{ip^2}{2m\hbar}(t-t')\right]\end{aligned} \tag{14}$$

由于

$$\mathrm{d}\vec{p} = \mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z \tag{15}$$

且 $\mathrm{d}p_x, \mathrm{d}p_y, \mathrm{d}p_z$ 等价

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}p_x\exp\left[\frac{i}{\hbar}p_x(x-x')-\frac{i}{2m\hbar}p_x^2(t-t')\right] \\ =&\int_{-\infty}^{\infty}\mathrm{d}p_x\exp\left\{\left[-\frac{i}{2m\hbar}(t-t')\right]\left(p_x^2-2mp_x\frac{x-x'}{t-t'}\right)\right\} \\ =&\int_{-\infty}^{\infty}\mathrm{d}p_x\exp\left\{\left[-\frac{i}{2m\hbar}(t-t')\right]\left[\left(p_x-m\frac{x-x'}{t-t'}\right)^2-m^2\frac{(x-x')^2}{(t-t')^2}\right]\right\} \\ =&\exp\left[\frac{im(x-x')^2}{2\hbar(t-t')}\right]\int_{-\infty}^{\infty}\mathrm{d}p_x\exp\left\{\left[-\frac{i}{2m\hbar}(t-t')\right]\left[\left(p_x-m\frac{x-x'}{t-t'}\right)^2\right]\right\} \\ =&\exp\left[\frac{im(x-x')^2}{2\hbar(t-t')}\right]\int_{-\infty}^{\infty}\mathrm{d}p_x\exp\left[-\frac{i}{2m\hbar}(t-t')p_x^2\right]\end{aligned} \tag{16}$$

根据 Euler integral

$$\int_{-\infty}^{\infty}\mathrm{d}xe^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \tag{17}$$

$$\int_{-\infty}^{\infty}\mathrm{d}p_x\exp\left[\frac{i}{\hbar}p_x(x-x')-\frac{i}{2m\hbar}p_x^2(t-t')\right] = \exp\left[\frac{im(x-x')^2}{2\hbar(t-t')}\right]\sqrt{\frac{2m\pi\hbar}{i(t-t')}} \tag{18}$$

故

$$K(\vec{r}, t; \vec{r}', t') = \left[\cancel{\frac{2m\pi\hbar}{i(t-t')}}\right]^{\frac{3}{2}} \exp\left[\frac{im(\vec{r}-\vec{r}')^2}{2\hbar(t-t')}\right] \tag{19}$$

$$\left(\frac{m}{2\pi i\hbar(t_b-t_a)}\right)^{\frac{3}{2}} \exp\left[\frac{i}{\hbar}\frac{m}{2}\frac{(x_b-x_a)^2}{t_b-t_a}\right]$$

General Case

回到前面不含时情况下的波函数

$$\begin{aligned}
\psi(\vec{r}, t) &= \int \mathrm{d}\vec{r}' \langle\vec{r}| \exp\left[-\frac{i}{\hbar}H(t-t')\right] |\vec{r}'\rangle \psi(\vec{r}', t') \\
&= \int \mathrm{d}\vec{r}' \langle\vec{r}| \exp\left[-\frac{i}{\hbar}H(t-t''+t''-t')\right] |\vec{r}'\rangle \psi(\vec{r}', t') \\
&= \int \mathrm{d}\vec{r}' \langle\vec{r}| \exp\left[-\frac{i}{\hbar}H(t-t'')\right] \exp\left[-\frac{i}{\hbar}H(t''-t')\right] |\vec{r}'\rangle \psi(\vec{r}', t') \\
&= \int \mathrm{d}\vec{r}' \int \mathrm{d}\vec{r}'' \langle\vec{r}| \exp\left[-\frac{i}{\hbar}H(t-t'')\right] |\vec{r}''\rangle \langle\vec{r}''| \exp\left[-\frac{i}{\hbar}H(t''-t')\right] |\vec{r}'\rangle \psi(\vec{r}', t') \\
&= \int \mathrm{d}\vec{r}' \int \mathrm{d}\vec{r}'' K(\vec{r}, t; \vec{r}'', t'') K(\vec{r}'', t''; \vec{r}', t') \psi(\vec{r}', t')
\end{aligned} \tag{20}$$

又

$$\psi(\vec{r}, t) = \int \mathrm{d}\vec{r}' K(\vec{r}, t; \vec{r}', t') \psi(\vec{r}', t') \tag{21}$$

即

$$K(\vec{r}, t; \vec{r}', t') = \int \mathrm{d}\vec{r}'' K(\vec{r}, t; \vec{r}'', t'') K(\vec{r}'', t''; \vec{r}', t') \tag{22}$$

因此

$$K(\vec{r}, t; \vec{r}', t') = \int \mathrm{d}\vec{r}_1 \int \mathrm{d}\vec{r}_2 \cdots \int \mathrm{d}\vec{r}_{N-1} K(\vec{r}, t; \vec{r}_{N-1}, t_{N-1}) K(\vec{r}_{N-1}, t_{N-}; \vec{r}_{N-2}, t_{N-2}) \cdots K(\vec{r}_1, t_1; \vec{r}', t') \tag{23}$$

2 Equation of Motion for $K(\vec{r}, t; \vec{r}', t')$

前面我们讨论的都是比较熟悉的薛定谔方程里面的东西，接下来我们来研究 $K(\vec{r}, t; \vec{r}', t')$ 的另一个性质，即讨论格林函数 $K(\vec{r}, t; \vec{r}', t')$ 的运动方程。

$$\psi(\vec{r}, t) = \int K(\vec{r}, t; \vec{r}', t') \psi(\vec{r}', t') \mathrm{d}\vec{r}' \tag{24}$$

将 $\left(i\hbar\frac{\partial}{\partial t} - H\right)$ 算符作用在方程两边

$$0 = \int \left[\left(i\hbar\frac{\partial}{\partial t} - H\right) K(\vec{r}, t; \vec{r}', t')\right] \psi(\vec{r}', t') \mathrm{d}\vec{r}' \tag{25}$$

因此我们得到 K 的运动方程

$$\left(i\hbar\frac{\partial}{\partial t} - H\right) K(\vec{r}, t; \vec{r}', t') = 0 \tag{26}$$

传播子代表是 t' 时刻对 t 时刻的影响，也就是说 t 时刻的性质是由 t' 时刻决定的，且 $t > t'$。当 $t < t'$ 时，若 $K \neq 0$，则代表后面时刻 t' 可以影响前面时刻 t，这显然是不对的。因此

$$\begin{cases} \left(i\hbar\frac{\partial}{\partial t} - H\right) K(\vec{r}, t; \vec{r}', t') = 0 & (t > t') \\ K(\vec{r}, t; \vec{r}', t') = 0 & (t < t') \end{cases} \tag{27}$$

在 $t=t'$ 时，有奇性 (singular)，即

$$\left(i\hbar\frac{\partial}{\partial t}-H\right)K(\vec{r},t;\vec{r}',t')=c(\vec{r},\vec{r}')\delta(t-t') \tag{28}$$

接下来我们来确定 $c(\vec{r},\vec{r}')$，两边积分

$$\int_{-\infty}^{\infty}\left(i\hbar\frac{\partial}{\partial t}-H\right)K(\vec{r},t;\vec{r}',t')\mathrm{d}t=\int_{-\infty}^{\infty}c(\vec{r},\vec{r}')\delta(t-t')\mathrm{d}t \tag{29}$$

$$\begin{aligned}\mathrm{LHS}=&\int_{-\infty}^{t'-}\left(i\hbar\frac{\partial}{\partial t}-H\right)K(\vec{r},t;\vec{r}',t')\mathrm{d}t+\int_{t'-}^{t'+}\left(i\hbar\frac{\partial}{\partial t}-H\right)K(\vec{r},t;\vec{r}',t')\mathrm{d}t\\&+\int_{t'+}^{\infty}\left(i\hbar\frac{\partial}{\partial t}-H\right)K(\vec{r},t;\vec{r}',t')\mathrm{d}t\\=&\int_{t'-}^{t'+}\left(i\hbar\frac{\partial}{\partial t}-H\right)K(\vec{r},t;\vec{r}',t')\mathrm{d}t=\int_{t'-}^{t'+}i\hbar\frac{\partial}{\partial t}K(\vec{r},t;\vec{r}',t')\mathrm{d}t\\=&\ i\hbar K(\vec{r},t=t';\vec{r}',t')\end{aligned} \tag{30}$$

当 H 不含时时，$K(\vec{r},t=t';\vec{r}',t')=\delta(\vec{r}-\vec{r}')$，因此

$$\mathrm{LHS}=i\hbar\delta(\vec{r}-\vec{r}') \tag{31}$$

$$\mathrm{RHS}=\int_{-\infty}^{\infty}c(\vec{r},\vec{r}')\delta(t-t')\mathrm{d}t=c(\vec{r},\vec{r}') \tag{32}$$

由 LHS=RHS 得

$$c(\vec{r},\vec{r}')=i\hbar\delta(\vec{r}-\vec{r}') \tag{33}$$

则

$$\left(i\hbar\frac{\partial}{\partial t}-H\right)K(\vec{r},t;\vec{r}',t')=i\hbar\delta(\vec{r}-\vec{r}')\delta(t-t') \tag{34}$$

这就是 $K(\vec{r},t;\vec{r}',t')$ 的运动方程。

3 Time-dependent Case

前面我们讨论的都是 H 不含时的情况，接下来我们讨论 H 含时的情况

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle=H(t)|\psi(t)\rangle \tag{35}$$

可以将波函数写成以下形式

$$|\psi(t)\rangle=U(t,t')|\psi(t')\rangle \tag{36}$$

$U(t,t')$ 为演化算符 (evolution operator)，将 $|\psi(t)\rangle$ 代回薛定谔方程得到

$$i\hbar\frac{\partial}{\partial t}U(t,t')=H(t)U(t,t') \tag{37}$$

对波函数作用 $\langle\vec{r}|$

$$\langle\vec{r}|\psi(t)\rangle=\langle\vec{r}|U(t,t')|\psi(t')\rangle=\int\mathrm{d}\vec{r}'\langle\vec{r}|U(t,t')|\vec{r}'\rangle\langle\vec{r}'|\psi(t')\rangle \tag{38}$$

即

$$\psi(\vec{r},t)=\int\mathrm{d}\vec{r}'K(\vec{r},t;\vec{r}',t')\psi(\vec{r}',t') \tag{39}$$

定义普遍情况的传播子

$$K(\vec{r},t;\vec{r}',t') = \langle\vec{r}|\,U(t,t')\,|\vec{r}'\rangle \tag{40}$$

不断重复作用演化算符，可以得到

$$U(t,t') = U(t,t_{N-1})U(t_{N-1},t_{N-2})\cdots U(t_1,t') \tag{41}$$

因此

$$\begin{aligned}K(\vec{r},t;\vec{r}',t') &= \langle\vec{r}|\,U(t,t')\,|\vec{r}'\rangle = \langle\vec{r}|\,U(t,t_{N-1})U(t_{N-1},t_{N-2})\cdots U(t_1,t')\,|\vec{r}'\rangle \\ &= \int \mathrm{d}\vec{r}_{N-1}\int \mathrm{d}\vec{r}_{N-2}\cdots\int \mathrm{d}\vec{r}_1\,\langle\vec{r}|\,U(t,t_{N-1})\,|\vec{r}_{N-1}\rangle\,\langle\vec{r}_{N-1}|\,U(t_{N-1},t_{N-2})\,|\vec{r}_{N-2}\rangle\cdots\langle\vec{r}_1|\,U(t_1,t')\,|\vec{r}'\rangle \\ &= \int \mathrm{d}\vec{r}_{N-1}\int \mathrm{d}\vec{r}_{N-2}\cdots\int \mathrm{d}\vec{r}_1 K(\vec{r},t;\vec{r}_{N-1},t_{N-1})K(\vec{r}_{N-1},t_{N-1};\vec{r}_{N-2},t_{N-2})\cdots K(\vec{r}_1,t_1;\vec{r}',t')\end{aligned} \tag{42}$$

很容易发现并验证

$$U^\dagger(t,t') = U(t,t') \tag{43}$$

$$-i\hbar\frac{\partial}{\partial t}U^\dagger(t,t') = H(t)U^\dagger(t,t') \tag{44}$$

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(t')|\,U^\dagger(t,t')U(t,t')\,|\psi(t')\rangle = 1 \tag{45}$$

因此

$$U^\dagger(t,t')U(t,t') = 1 \tag{46}$$

$$U^\dagger(t,t') = U^{-1}(t,t') = U(t,t') \tag{47}$$

因此 U 是幺正算符 (unitary operator)

定义

$$|\vec{r},t\rangle = U^\dagger(t,0)\,|\vec{r}\rangle \tag{48}$$

则

$$\langle\vec{r},t| = \langle\vec{r}|\,U(t,0) \tag{49}$$

$$\begin{aligned}K(\vec{r},t;\vec{r}',t') &= \langle\vec{r}|\,U(t,t')\,|\vec{r}'\rangle = \langle\vec{r}|\,U(t,0)U(0,t')\,|\vec{r}'\rangle \\ &= \langle\vec{r}|\,U(t,0)U^\dagger(0,t')\,|\vec{r}'\rangle = \langle\vec{r},t|\vec{r}',t'\rangle\end{aligned} \tag{50}$$

接下来讨论特殊情况：H(t)=H

$$U(t,t') = \exp\left[-\frac{iH(t-t')}{\hbar}\right] \tag{51}$$

$$U(t,0) = \exp\left(-\frac{iHt}{\hbar}\right) \tag{52}$$

$$\begin{aligned}|\vec{r},t\rangle &= U^\dagger(t,0)\,|\vec{r}\rangle = \exp\left(\frac{iHt}{\hbar}\right)|\vec{r}\rangle = \sum_n \exp\left(\frac{iHt}{\hbar}\right)|n\rangle\,\langle n|\vec{r}\rangle \\ &= \sum_n \exp\left(\frac{iHt}{\hbar}\right)|n\rangle\,\psi_n^\dagger(\vec{r}) = \sum_n |n\rangle\exp\left(\frac{iE_nt}{\hbar}\right)\psi_n^\dagger(\vec{r})\end{aligned} \tag{53}$$

$$\begin{aligned}K(\vec{r},t;\vec{r}',t') &= \langle\vec{r},t|\vec{r}',t'\rangle = \sum_n\sum_m \langle m|\exp\left(-\frac{iE_mt}{\hbar}\right)\psi_m(\vec{r})\exp\left(\frac{iE_nt'}{\hbar}\right)\psi_n^\dagger(\vec{r}')\,|n\rangle \\ &= \sum_n \exp\left[\frac{iE_n(t'-t)}{\hbar}\right]\psi_n(\vec{r})\psi_n^\dagger(\vec{r}') = \sum_n \psi_n(\vec{r},t)\psi_n^\dagger(\vec{r}',t')\end{aligned} \tag{54}$$

与第一节中哈密顿量不含时定义推导得到的传播子一致。

我们来证明一个单位算符

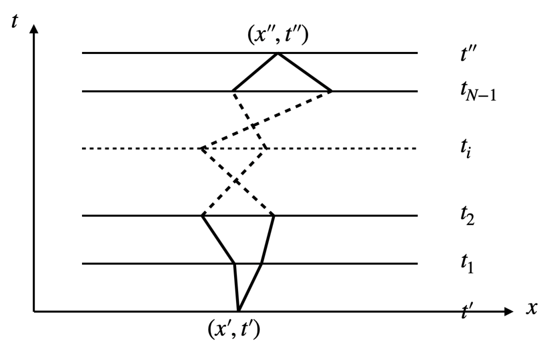
$$\int \mathrm{d}\vec{r}\,|\vec{r},t\rangle\langle\vec{r},t| = \int \mathrm{d}\vec{r}e^{iHt/\hbar}|\vec{r}\rangle\langle\vec{r}|e^{-iHt/\hbar} = e^{iHt/\hbar}e^{-iHt/\hbar} = 1 \tag{55}$$

$$K(\vec{r},t;\vec{r}',t') = \langle\vec{r},t|\vec{r}',t'\rangle = \int \mathrm{d}\vec{r}''\,\langle\vec{r},t|\vec{r}'',t''\rangle\langle\vec{r}'',t''|\vec{r}',t'\rangle \tag{56}$$

4 Feynman's Formulation of Quantum Mechanics

为了书写方便，我们写成一维形式

$$\begin{aligned}&\langle x'',t''|x',t'\rangle\\&=\int \mathrm{d}x_{N-1}\int \mathrm{d}x_{N-2}\cdots\int \mathrm{d}x_1\,\langle x'',t''|x_{N-1},t_{N-1}\rangle\langle x_{N-1},t_{N-1}|x_{N-2},t_{N-2}\rangle\cdots\langle x_1,t_1|x',t'\rangle\end{aligned} \tag{57}$$



粒子从 (x',t') 以任意一条路径运动到 (x'',t'')

Dirac's Remark

若 $t_2 \to t_1$

$$\langle x_2,t_2|x_1,t_1\rangle \sim \exp\left[\frac{i\int_{t_1}^{t_2} L_{\text{classical}}(x,\dot{x},t)\mathrm{d}t}{\hbar}\right] \tag{58}$$

拉格朗日作用量

$$S(n,n-1) = \int_{t_{n-1}}^{t_n} L_{\text{classical}}(x,\dot{x},t)\mathrm{d}t \tag{59}$$

$$\langle x_n,t_n|x_{n-1},t_{n-1}\rangle \sim \exp\left[\frac{iS(n,n-1)}{\hbar}\right] \tag{60}$$

设 $\Delta t = t_n - t_{n-1} \to 0$

$$\begin{aligned}S(n,n-1) &= \int_{t_{n-1}}^{t_n}\left[\frac{m\dot{x}^2}{2} - V(x)\right]\mathrm{d}t\\&= \Delta t\left[\frac{m}{2}\left(\frac{x_n - x_{n-1}}{\Delta t}\right)^2 - V\left(\frac{x_n + x_{n-1}}{2}\right)\right]\end{aligned} \tag{61}$$

我们可以在 $\langle x'', t''|x', t'\rangle$ 直接放入无穷个积分，使 Δt 无穷小

$$
\begin{aligned}
&\langle x'', t''|x', t'\rangle \\
=&\int \mathrm{d}x_{N-1}\int \mathrm{d}x_{N-2}\cdots\int \mathrm{d}x_1\, \langle x'', t''|x_{N-1}, t_{N-1}\rangle \langle x_{N-1}, t_{N-1}|x_{N-2}, t_{N-2}\rangle\cdots\langle x_1, t_1|x', t'\rangle \\
\sim&\int \mathrm{d}x_{N-1}\int \mathrm{d}x_{N-2}\cdots\int \mathrm{d}x_1 \prod_{n=1}^{N}\exp\left[\frac{i}{\hbar}S(n,n-1)\right] \\
=&\int D[x(t)]\exp\left[\frac{i}{\hbar}\sum_{n=1}^{N}S(n,n-1)\right] \\
=&\int D[x(t)]\exp\left\{\frac{i}{\hbar}S_N[x(t)]\right\}
\end{aligned}
\tag{62}
$$

其中

$$S_N[x(t)] = \sum_{n=1}^{N}S(n,n-1) = \int_{t'}^{t''} L(x,\dot{x},t)\mathrm{d}t \tag{63}$$

设 $\vec{r}(t_0)=\vec{r}', \vec{r}(t_N)=\vec{r}'', t_0=t', t_N=t''$，且 $t_0, t_1, \cdots, t_{N-1}, t_N$ 等分

$$\varepsilon = t_j - t_{j-1} \qquad j=1,2,\cdots,N \tag{64}$$

$$S_N[\vec{r}(t)] = \int_{t'}^{t''} L(\vec{r},\dot{\vec{r}},t)\mathrm{d}t = \varepsilon\sum_{j=1}^{N} L\left(\frac{\vec{r}_j+\vec{r}_{j-1}}{2}, \frac{\vec{r}_j-\vec{r}_{j-1}}{\varepsilon}, t'+j\varepsilon\right) \tag{65}$$

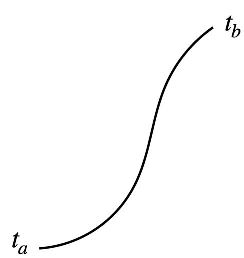
$$K_N(\vec{r}'', t''; \vec{r}', t') = \langle \vec{r}'', t''|\vec{r}', t'\rangle_N \sim \int D[\vec{r}(t)]\exp\left\{\frac{i}{\hbar}S_N[\vec{r}(t)]\right\} \tag{66}$$

我们希望

$$K(\vec{r}'', t''; \vec{r}', t') = \lim_{N\to\infty} K_N(\vec{r}'', t''; \vec{r}', t') \tag{67}$$

Review: Euler-Lagrange Principle

已知一个粒子 t_a 和 t_b 时刻的位置，它的运动轨迹是使作用量 $S[x(t)]$ 最小的那一条。



根据最小作用量原理

$$
\begin{aligned}
\delta S[x(t)] &= \delta\int_{t_a}^{t_b} L(x,\dot{x},t)\mathrm{d}t \\
&= \int_{t_a}^{t_b}\left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta\dot{x}\right)\mathrm{d}t \\
&= \int_{t_a}^{t_b}\left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\frac{\mathrm{d}}{\mathrm{d}t}\delta x\right)\mathrm{d}t \\
&= \int_{t_a}^{t_b}\left[\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\delta x\mathrm{d}t + \left[\frac{\partial L}{\partial \dot{x}}\delta x\right]_{t_a}^{t_b} \\
&= \int_{t_a}^{t_b}\left[\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\delta x\mathrm{d}t = 0
\end{aligned}
\tag{68}
$$

得到 Euler-Lagrange Equation

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \tag{69}$$

Example: 1-D Free Particle

$$L = \frac{1}{2}m\dot{x}^2 \tag{70}$$

代入拉格朗日方程

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \tag{71}$$

得

$$m\dot{x} = \text{constant} \tag{72}$$

$$p = m\dot{x} = m\frac{x'' - x'}{t'' - t'} \tag{73}$$

$$S_N[x(t)] = \int_{t'}^{t''} L(x, \dot{x}, t)\mathrm{d}t = \frac{1}{2}\int_{t'}^{t''} m\dot{x}^2\mathrm{d}t = \frac{m}{2}\frac{(x'' - x')^2}{t'' - t'} \tag{74}$$

$$S(x_{j+1}, t_{j+1}; x_j, t_j) = \frac{m}{2}\frac{(x_{j+1} - x_j)^2}{t_{j+1} - t_j} = \frac{m}{2}\frac{(x_{j+1} - x_j)^2}{\varepsilon} \tag{75}$$

由 Dirac's remark 我们知道，当 $t \to t'$ 时

$$K(x, t; x', t') = C\exp\left[\frac{iS(x, t; x', t')}{\hbar}\right] = C\exp\left[\frac{im}{2\hbar}\frac{(x - x')^2}{t - t'}\right] \tag{76}$$

接下来我们用初始条件来定 C。当 $t = t'$ 时，

$$K(x, t; x', t') = \delta(x - x') \tag{77}$$

已知积分

$$\lim_{\alpha\to\infty} e^{-i\alpha x^2}\sqrt{\frac{\alpha}{\pi}} = e^{-\frac{i\pi}{4}}\delta(x) \tag{78}$$

$$\begin{aligned}
\lim_{t\to t'} K(x, t; x', t') &= C\lim_{t\to t'}\exp\left[\frac{im}{2\hbar}\frac{(x - x')^2}{t - t'}\right] \\
&= C\lim_{t\to t'}\exp\left[\frac{im}{2\hbar}\frac{(x - x')^2}{t - t'}\right]\sqrt{-\frac{m(x - x')^2}{2\pi\hbar(t - t')}}\sqrt{-\frac{2\pi\hbar(t - t')}{m(x - x')^2}} \\
&= Ce^{-i\frac{\pi}{4}}\sqrt{-\frac{2\pi\hbar(t - t')}{m(x - x')^2}}\delta(x - x') = \delta(x - x')
\end{aligned} \tag{79}$$

$$C = \sqrt{-\frac{m}{2\pi\hbar(t - t')}}e^{i\frac{\pi}{4}} = \sqrt{-\frac{m}{2\pi\hbar(t - t')}e^{i\frac{\pi}{2}}} = \sqrt{\frac{m}{2\pi i\hbar(t - t')}} \tag{80}$$

$$K(x, t; x', t') = C\exp\left[\frac{im}{2\hbar}\frac{(x - x')^2}{t - t'}\right] = \sqrt{\frac{m}{2\pi i\hbar(t - t')}}\exp\left[\frac{im}{2\hbar}\frac{(x - x')^2}{t - t'}\right] \tag{81}$$

当 $(t - t')$ 为有限大小时，我们将 t‘ 到 t 分成无穷等份

$$\begin{aligned}
K(x, t; x', t') &= \int \mathrm{d}x_1\int \mathrm{d}x_2\cdots\int \mathrm{d}x_{N-1}\left[\frac{m}{2\pi i\hbar(t - t')}\right]^{\frac{N}{2}}\exp\left[\frac{im}{2\hbar\varepsilon}\sum_{j=0}^{N-1}(x_{j+1} - x_j)^2\right] \\
&= \left[\frac{m}{2\pi i\hbar(t - t')}\right]^{\frac{N}{2}}\int \mathrm{d}x_1\int \mathrm{d}x_2\cdots\int \mathrm{d}x_{N-1}\exp\left[\frac{im}{2\hbar\varepsilon}\sum_{j=0}^{N-1}(x_{j+1} - x_j)^2\right]
\end{aligned} \tag{82}$$

根据积分

$$\int_{-\infty}^{\infty}\mathrm{d}x_2\exp\left[\alpha(x_1-x_2)^2+\beta(x_3-x_2)^2\right]=\exp\left[\frac{\alpha\beta}{\alpha+\beta}(x_1-x_3)^2\right]\sqrt{-\frac{\pi}{\alpha+\beta}} \tag{83}$$

于是

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}x_1\int_{-\infty}^{\infty}\mathrm{d}x_2\cdots\int_{-\infty}^{\infty}\mathrm{d}x_{N-1}\exp\left[\frac{im}{2\hbar\varepsilon}\sum_{j=0}^{N-1}(x_{j+1}-x_j)^2\right]\\&=\left(\frac{2\hbar}{im}\right)^{\frac{N-1}{2}}\int_{-\infty}^{\infty}\mathrm{d}y_1\int_{-\infty}^{\infty}\mathrm{d}y_2\cdots\int_{-\infty}^{\infty}\mathrm{d}y_{N-1}\exp\left[\frac{1}{\varepsilon}\sum_{j=0}^{N-1}(y_{j+1}-y_j)^2\right]\end{aligned} \tag{84}$$

其中 $y=\sqrt{\frac{im}{2\hbar}}x$。依次积分，设

$$\alpha_1=\frac{1}{\varepsilon}\qquad\beta=\frac{1}{\varepsilon}\qquad\alpha_2=\frac{\alpha_1\beta}{\alpha_1+\beta}=\frac{1}{2\varepsilon}\qquad\alpha_1+\beta=\frac{2}{\varepsilon} \tag{85}$$

猜测

$$\alpha_m=\frac{1}{m\varepsilon} \tag{86}$$

则

$$\alpha_{m+1}=\frac{\alpha_m\beta}{\alpha_m+\beta}=\frac{1}{(m+1)\varepsilon} \tag{87}$$

得证。得到普遍表达式

$$\alpha_m+\beta=\frac{1}{m\varepsilon}+\frac{1}{\varepsilon}=\frac{m+1}{m\varepsilon} \tag{88}$$

于是

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}x_1\int_{-\infty}^{\infty}\mathrm{d}x_2\cdots\int_{-\infty}^{\infty}\mathrm{d}x_{N-1}\exp\left[\frac{im}{2\hbar\varepsilon}\sum_{j=0}^{N-1}(x_{j+1}-x_j)^2\right]\\&=\left(\frac{2\hbar}{im}\right)^{\frac{N-1}{2}}\int_{-\infty}^{\infty}\mathrm{d}y_1\int_{-\infty}^{\infty}\mathrm{d}y_2\cdots\int_{-\infty}^{\infty}\mathrm{d}y_{N-1}\exp\left[\frac{1}{\varepsilon}\sum_{j=0}^{N-1}(y_{j+1}-y_j)^2\right]\\&=\left(\frac{2\hbar}{im}\right)^{\frac{N-1}{2}}\exp\left[\frac{1}{N\varepsilon}(y-y')^2\right]\sqrt{-\frac{\pi\varepsilon}{2}}\sqrt{-\frac{\pi2\varepsilon}{3}}\cdots\sqrt{-\frac{\pi m\varepsilon}{m+1}}\cdots\sqrt{-\frac{\pi(N-1)\varepsilon}{N}}\\&=\left(\frac{2\hbar}{im}\right)^{\frac{N-1}{2}}\exp\left[\frac{im}{2\hbar}\frac{(x-x')^2}{N\varepsilon}\right](-\pi\varepsilon)^{\frac{N-1}{2}}\frac{1}{\sqrt{N}}\\&=\exp\left[\frac{im}{2\hbar}\frac{(x-x')^2}{N\varepsilon}\right]\left(-\frac{2\pi\hbar\varepsilon}{im}\right)^{\frac{N-1}{2}}\frac{1}{\sqrt{N}}\end{aligned} \tag{89}$$

整理成

$$\begin{aligned}K(x,t;x',t')&=\left[\frac{m}{2\pi i\hbar(t-t')}\right]^{\frac{N}{2}}\int\mathrm{d}x_1\int\mathrm{d}x_2\cdots\int\mathrm{d}x_{N-1}\exp\left[\frac{im}{2\hbar\varepsilon}\sum_{j=0}^{N-1}(x_{j+1}-x_j)^2\right]\\&=\left(\frac{m}{2\pi i\hbar N\varepsilon}\right)^{\frac{N}{2}}\exp\left[\frac{im}{2\hbar}\frac{(x-x')^2}{N(t-t')}\right]\left(-\frac{2\pi\hbar\varepsilon}{im}\right)^{\frac{N-1}{2}}\frac{1}{\sqrt{N}}\\&=\sqrt{\frac{m}{2\pi i\hbar(t-t')}}\exp\left[\frac{im}{2\hbar}\frac{(x-x')^2}{N(t-t')}\right]\end{aligned} \tag{90}$$

我们发现，用经典理论导出的结果与量子力学的结果完全一致，量子力学是可以从经典理论中导出的。但量子力学和经典力学的过程完全不同，如量子力学中粒子没有固定路线、算符之间的不对易性、海森堡不确定性原理等，都与经典力学完全不一样。过程中发生了什么使得我们通过经典力学的作用量得到量子力学的结果?

回到一维自由粒子情况

$$K(x,t;x',t') = C\int_{-\infty}^{\infty}\mathrm{d}x_1\int_{-\infty}^{\infty}\mathrm{d}x_2\cdots\int_{-\infty}^{\infty}\mathrm{d}x_{N-1}\exp\left[\frac{im}{2\hbar\varepsilon}\sum_{j=0}^{N-1}(x_{j+1}-x_j)^2\right] \tag{91}$$

从中拿出一个积分

$$\begin{aligned}&\int\mathrm{d}x_j\exp\left[\frac{im}{2\hbar\varepsilon}(x_j-x_{j-1})^2\right]\exp\left[\frac{im}{2\hbar\varepsilon}(x_{j+1}-x_j)^2\right]\\ =&\int\mathrm{d}x_j\exp\left[\frac{im}{2\hbar\varepsilon}(x_j^2-2x_jx_{j-1}+x_{j-1}^2)\right]\exp\left[\frac{im}{2\hbar\varepsilon}(x_{j+1}^2-2x_{j+1}x_j+x_j^2)\right]\\ =&\exp\left[\frac{im}{2\hbar\varepsilon}(x_{j-1}^2+x_{j+1}^2)\right]\exp\left[-\frac{im}{\hbar\varepsilon}\left(\frac{x_{j-1}-x_{j+1}}{2}\right)^2\right]\int_{-\infty}^{\infty}\mathrm{d}x_j\exp\left[\frac{im}{\hbar\varepsilon}\left(x_j-\frac{x_{j-1}+x_{j+1}}{2}\right)^2\right]\end{aligned} \tag{92}$$

已知积分

$$\lim_{\alpha\to\infty}e^{i\alpha x^2}\sqrt{\frac{\alpha}{\pi}} = e^{\frac{i\pi}{4}}\delta(x) \tag{93}$$

当 $\hbar\to 0$ 时，退化到经典理论，令 $\alpha=\frac{m}{\hbar\varepsilon}\to\infty$

$$\begin{aligned}\lim_{\hbar\to 0}\exp\left[\frac{im}{\hbar\varepsilon}\left(x_j-\frac{x_{j-1}+x_{j+1}}{2}\right)^2\right] &= \lim_{\alpha\to\infty}\exp\left[i\alpha\left(x_j-\frac{x_{j-1}+x_{j+1}}{2}\right)^2\right]\sqrt{\frac{\alpha}{\pi}}\sqrt{\frac{\pi}{\alpha}}\\ &=\sqrt{\frac{\pi\hbar\varepsilon}{m}}e^{\frac{i\pi}{4}}\delta\left(x_j-\frac{x_{j-1}+x_{j+1}}{2}\right)\end{aligned} \tag{94}$$

在经典理论中，积分退化为一点的贡献，自由粒子的运动路线是一条直线。

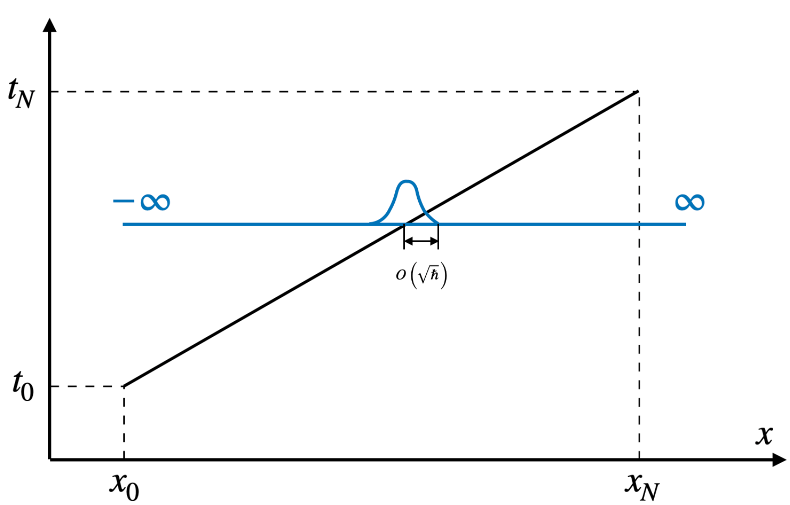
当 $\hbar$ 有限时，Gauss 函数有一个很小的贡献

$$\left(x_j-\frac{x_{j-1}+x_{j+1}}{2}\right)^2\frac{m}{\varepsilon}\frac{1}{\hbar}\sim O(1) \tag{95}$$

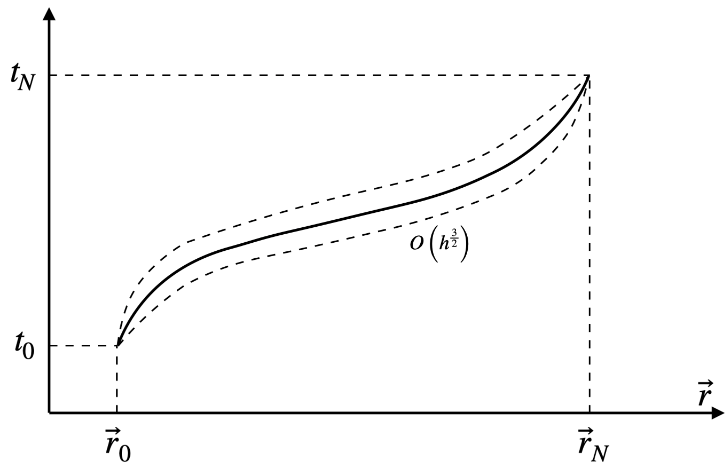
$$\left(x_j-\frac{x_{j-1}+x_{j+1}}{2}\right)^2\frac{m}{\varepsilon}\sim O(\hbar) \tag{96}$$

$$x_j\sim\frac{x_{j-1}+x_{j+1}}{2}+O(\sqrt{\hbar}) \tag{97}$$

展开宽度的量级是 $\sqrt{\hbar}$。



三维情况经典理论下粒子轨迹为一条曲线，而量子力学下有 $\hbar^{\frac{3}{2}}$ 的展开宽度。



5 从 Feynman 路径积分导出传播子

$$
\begin{aligned}
&K(\vec{r}',t';\vec{r},t)\\
&=\lim_{N\to\infty}\int\prod_{j=1}^{N-1}\mathrm{d}\vec{r}_j\exp\left\{\frac{i}{\hbar}S_N[\vec{r}(t)]\right\}\\
&=\lim_{N\to\infty}\int\prod_{j=1}^{N-1}\mathrm{d}\vec{r}_j\exp\left[\frac{i}{\hbar}\varepsilon\sum_{j=1}^{N}L\left(\frac{\vec{r}_j+\vec{r}_{j-1}}{2},\frac{\vec{r}_j-\vec{r}_{j-1}}{\varepsilon},t+j\varepsilon\right)\right]\\
&=\lim_{M\to\infty}\lim_{M-N\to\infty}\int\prod_{j=1}^{M-1}\mathrm{d}\vec{r}_j\prod_{i=M+1}^{N-1}\mathrm{d}\vec{r}_i\mathrm{d}\vec{r}_M\\
&\quad\exp\left\{\frac{i}{\hbar}\varepsilon\left[\sum_{j=1}^{M}L\left(\frac{\vec{r}_j+\vec{r}_{j-1}}{2},\frac{\vec{r}_j-\vec{r}_{j-1}}{\varepsilon},t+j\varepsilon\right)+\sum_{i=M+1}^{N}L\left(\frac{\vec{r}_i+\vec{r}_{i-1}}{2},\frac{\vec{r}_i-\vec{r}_{i-1}}{\varepsilon},t+M\varepsilon+i\varepsilon\right)\right]\right\}\\
&=\int\mathrm{d}\vec{r}_M\lim_{M\to\infty}\int\prod_{j=1}^{M-1}\mathrm{d}\vec{r}_j\exp\left[\frac{i}{\hbar}\varepsilon\sum_{j=1}^{M}L\left(\frac{\vec{r}_j+\vec{r}_{j-1}}{2},\frac{\vec{r}_j-\vec{r}_{j-1}}{\varepsilon},t+j\varepsilon\right)\right]\\
&\quad\lim_{N-M\to\infty}\int\prod_{i=1}^{M-1}\mathrm{d}\vec{r}_i\exp\left[\frac{i}{\hbar}\varepsilon\sum_{i=M}^{M}L\left(\frac{\vec{r}_i+\vec{r}_{i-1}}{2},\frac{\vec{r}_i-\vec{r}_{j-1}}{\varepsilon},t+M\varepsilon+i\varepsilon\right)\right]\\
&=\int\mathrm{d}\vec{r}_MK(\vec{r}',t';\vec{r}_M,t_M)K(\vec{r}_M,t_M;\vec{r},t)
\end{aligned}
\tag{98}
$$

$\vec{r}_M$ 是赝矢量 (dummy index)，用 $\vec{r}''$ 表示 $\vec{r}_M$

$$
K(\vec{r}',t';\vec{r},t)=\int\mathrm{d}\vec{r}''K(\vec{r}',t';\vec{r}'',t'')K(\vec{r}'',t'';\vec{r},t)
\tag{99}
$$

这个方程我们在薛定谔方程中提到过，现在我们从路径积分中导出同样的结果，验证了这一点。

6 Dirac's Remark ($t_2 \to t_1$)

$$
\begin{aligned}
\langle x_2, t_2 | x_1, t_1\rangle &\sim \exp\left[\frac{i}{\hbar}\int_{t_1}^{t_2} L(x,\dot{x},t)\mathrm{d}t\right] \\
&= \exp\left\{\frac{i}{\hbar}\left[\frac{1}{2}m\frac{(x_2-x_1)^2}{t_2-t_1} - (t_2-t_1)V(x_1)\right]\right\}
\end{aligned}
\tag{100}
$$

在量子力学中我们定义

$$
\begin{aligned}
\langle x_2, t_2 | x_1, t_1\rangle &= \langle x_2|\exp\left[-\frac{i}{\hbar}H(t_2-t_1)\right]|x_1\rangle \\
&= \langle x_2|\exp\left[-\frac{i}{\hbar}(H_0+V)(t_2-t_1)\right]|x_1\rangle
\end{aligned}
\tag{101}
$$

H_0 和 V 在量子力学中具有不对易性，利用如下关系式

$$
\exp[\varepsilon(A+B)] = \exp(\varepsilon A)\exp(\varepsilon B)\left[\exp\left(-\frac{1}{2}\varepsilon^2[A,B]\right) + O(\varepsilon^3)\right]
\tag{102}
$$

$$
\lim_{\varepsilon\to 0}\exp[\varepsilon(A+B)] = \exp(\varepsilon A)\exp(\varepsilon B)
\tag{103}
$$

由于 $(t_2 - t_1) \to 0$，

$$
\begin{aligned}
\langle x_2, t_2 | x_1, t_1\rangle &= \langle x_2|\exp\left[-\frac{i}{\hbar}H_0(t_2-t_1)\right]\exp\left[-\frac{i}{\hbar}V(t_2-t_1)\right]|x_1\rangle \\
&= \langle x_2|\exp\left[-\frac{i}{\hbar}H_0(t_2-t_1)\right]|x_1\rangle\exp\left[-\frac{i}{\hbar}V(x_1)(t_2-t_1)\right]
\end{aligned}
\tag{104}
$$

自由粒子的传播子

$$
\langle x_2|\exp\left[-\frac{i}{\hbar}H_0(t_2-t_1)\right]|x_1\rangle = \left[\frac{m}{2\pi i\hbar(t_2-t_1)}\right]^{\frac{1}{2}}\exp\left[\frac{im}{2\hbar}\frac{(x_2-x_1)^2}{t_2-t_1}\right]
\tag{105}
$$

$$
\langle x_2, t_2 | x_1, t_1\rangle = \left[\frac{m}{2\pi i\hbar(t_2-t_1)}\right]^{\frac{1}{2}}\exp\left\{\frac{i}{\hbar}\left[\frac{m}{2}\frac{(x_2-x_1)^2}{t_2-t_1} - (t_2-t_1)V(x_1)\right]\right\}
\tag{106}
$$

当 $t_2 \to t_1$ 时，Feynman 给出的结果和 Schrödinger 给出的结果一致。

7 从 Feynman 路径积分导出 $t'' - t' = \mathrm{finite}$ 时的传播子

$$
\begin{aligned}
\langle x'', t'' | x', t'\rangle &= \langle x''|\exp\left[-\frac{i}{\hbar}H(t''-t')\right]|x'\rangle \\
&= \langle x''|\exp\left[-\frac{i}{\hbar}H(t''-t_{N-1})\right]\cdots\exp\left[-\frac{i}{\hbar}H(t_{j+1}-t_j)\right]\cdots\exp\left[-\frac{i}{\hbar}H(t_1-t')\right]|x'\rangle \\
&= \int \mathrm{d}x_1\cdots\mathrm{d}x_{N-1}\,\langle x''|\exp\left[-\frac{i}{\hbar}H(t''-t_{N-1})\right]|x_{N-1}\rangle\cdots \\
&\quad \langle x_{j+1}|\exp\left[-\frac{i}{\hbar}H(t_{j+1}-t_j)\right]|x_j\rangle\cdots\langle x_1|\exp\left[-\frac{i}{\hbar}H(t_1-t')\right]|x'\rangle \\
&= \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{N}{2}}\int \mathrm{d}x_1\cdots\mathrm{d}x_{N-1}\exp\left\{\sum_{j=1}^{N-1}\left[\frac{im(x_{j+1}-x_j)^2}{2\hbar\varepsilon} - \frac{i}{\hbar}\varepsilon V(x_j)\right]\right\} \\
&= \int D[x(t)]\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - V(x)\right]\mathrm{d}t\right\}
\end{aligned}
\tag{107}
$$

根据 Lie-Trotter Formula

$$\exp[it(A+B)] = \lim_{N\to\infty}\left[\exp\left(\frac{itA}{N}\right)\exp\left(\frac{itB}{N}\right)\right]^N \tag{108}$$

$$\begin{aligned}
&\langle x|\exp\left[-\frac{i}{\hbar}H(t-t')\right]|x'\rangle \\
=&\langle x|\exp\left[-\frac{i}{\hbar}(H_0+V)(t-t')\right]|x'\rangle \\
=&\langle x|\left[\exp\left(-\frac{iH_0\varepsilon}{\hbar}\right)\exp\left(-\frac{iV\varepsilon}{\hbar}\right)\right]^N|x'\rangle \\
=&\int \mathrm{d}x_1\cdots\mathrm{d}x_{N-1}\,\langle x''|\exp\left(-\frac{iH_0\varepsilon}{\hbar}\right)\exp\left(-\frac{iV\varepsilon}{\hbar}\right)|x_{N-1}\rangle\cdots\langle x_1|\exp\left(-\frac{iH_0\varepsilon}{\hbar}\right)\exp\left(-\frac{iV\varepsilon}{\hbar}\right)|x'\rangle \\
=&\langle x_N=x|\prod_{j=1}^{N-1}\int_{-\infty}^{\infty}\mathrm{d}x_j\exp\left(-\frac{i\varepsilon}{2m\hbar}p^2\right)\exp\left[-\frac{i\varepsilon}{\hbar}V(x)\right]|x_{N-1}\rangle\langle x_{N-1}|\exp\left(-\frac{i\varepsilon}{2m\hbar}p^2\right) \\
&\exp\left[-\frac{i\varepsilon}{\hbar}V(x)\right]|x_{N-2}\rangle\cdots|x_1\rangle\langle x_1|\exp\left(-\frac{i\varepsilon}{2m\hbar}p^2\right)\exp\left[-\frac{i\varepsilon}{\hbar}V(x)\right]|x_1=x'\rangle \\
=&\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{N}{2}}\prod_{j=1}^{N-1}\int_{-\infty}^{\infty}\mathrm{d}x_j\exp\left[\frac{i}{\hbar}\int_{t'}^{t}L\mathrm{d}t''\right]
\end{aligned} \tag{109}$$

$$K(x,t,x',t') = \int D[x(t)]\exp\left[\frac{i}{\hbar}S(t.t')\right] \tag{110}$$

我们可以从 Feynman 路径积分给出量子力学的结果。

8 从 Feynman 路径积分导出传播子的运动方程

已知运动方程

$$i\hbar\frac{\partial}{\partial t}\langle x,t|x',t'\rangle = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\langle x,t|x',t'\rangle \tag{111}$$

$$\begin{aligned}
\langle x,t|x',t'\rangle \sim& \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{N}{2}}\prod_{j=1}^{N-1}\int\mathrm{d}x_j\exp\left[\frac{i}{\hbar}S(t,t')\right] \\
=&\left[\frac{m}{2\pi i\hbar(t-t_{N-1})}\right]^{\frac{1}{2}}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{N-1}{2}}\prod_{j=1}^{N-1}\int\mathrm{d}x_j\exp\left[\frac{i}{\hbar}S(t,t_{N-1})\right]\prod_{n=1}^{N-1}\exp\left[\frac{i}{\hbar}S(n,n-1)\right]
\end{aligned} \tag{112}$$

方程左边

$$i\hbar\frac{\partial}{\partial t}\langle x,t|x',t'\rangle = i\hbar\left(-\frac{1}{2}\right)\frac{1}{t-t_{N-1}}\langle x,t|x',t'\rangle - \left[\frac{\partial}{\partial t}S(t,t_{N-1})\right]\langle x,t|x',t'\rangle \tag{113}$$

其中

$$\begin{aligned}
\frac{\partial}{\partial t}S(t,t_{N-1}) =& \frac{\partial}{\partial t}\left[(t-t_{N-1})L\left(\frac{x+x_{N-1}}{2},\frac{x-x_{N-1}}{t-t_{N-1}},\frac{t+t_{N-1}}{2}\right)\right] \\
=&\frac{\partial}{\partial t}\left\{(t-t_{N-1})\left[\frac{1}{2}m\left(\frac{x-x_{N-1}}{t-t_{N-1}}\right)^2-V\left(\frac{x+x_{N-1}}{2}\right)\right]\right\} \\
=&-\frac{1}{2}m\left(\frac{x-x_{N-1}}{t-t_{N-1}}\right)^2-V\left(\frac{x+x_{N-1}}{2}\right) \\
=&-T-V
\end{aligned} \tag{114}$$

代回 Eq.(113)

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\left\langle x,t|x',t'\right\rangle =& i\hbar\left(-\frac{1}{2}\right)\frac{1}{t-t_{N-1}}\left\langle x,t|x',t'\right\rangle-\left[\frac{\partial}{\partial t}S(t,t_{N-1})\right]\left\langle x,t|x',t'\right\rangle \\ =& \left[i\hbar\left(-\frac{1}{2}\right)\frac{1}{t-t_{N-1}}+\frac{1}{2}m\left(\frac{x-x_{N-1}}{t-t_{N-1}}\right)^2+V\left(x\right)\right]\left\langle x,t|x',t'\right\rangle \end{aligned} \tag{115}$$

方程右边

$$\text{RHS}=\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\left\langle x,t|x',t'\right\rangle \tag{116}$$

其中

$$\begin{aligned} \frac{\partial}{\partial x}\left\langle x,t|x',t'\right\rangle =& \frac{i}{\hbar}\left\langle x,t|x',t'\right\rangle\frac{\partial}{\partial x}S(t,t_{N-1}) \\ =& \frac{i}{\hbar}\left\langle x,t|x',t'\right\rangle\frac{\partial}{\partial x}\left\{(t-t_{N-1})\left[\frac{1}{2}m\left(\frac{x-x_{N-1}}{t-t_{N-1}}\right)^2-V\left(\frac{x+x_{N-1}}{2}\right)\right]\right\} \\ =& \frac{im}{\hbar}\frac{x-x_{N-1}}{t-t_{N-1}}\left\langle x,t|x',t'\right\rangle \end{aligned} \tag{117}$$

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\left\langle x,t|x',t'\right\rangle =& \frac{\partial}{\partial x}\left(\frac{im}{\hbar}\frac{x-x_{N-1}}{t-t_{N-1}}\left\langle x,t|x',t'\right\rangle\right) \\ =& \left[\frac{im}{\hbar}\frac{1}{t-t_{N-1}}-\frac{m^2}{\hbar^2}\left(\frac{x-x_{N-1}}{t-t_{N-1}}\right)^2\right]\left\langle x,t|x',t'\right\rangle \end{aligned} \tag{118}$$

$$\begin{aligned} \text{RHS} =& \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\left\langle x,t|x',t'\right\rangle \\ =& \left[-\frac{i\hbar}{2}\frac{1}{t-t_{N-1}}+\frac{m}{2}\left(\frac{x-x_{N-1}}{t-t_{N-1}}\right)^2+V(x)\right]\left\langle x,t|x',t'\right\rangle \end{aligned} \tag{119}$$

LHS=RHS，故运动方程得证。即可以通过 Feynman 路径积分导出运动方程。

9 Equivalence of Feynman's Formulation and Schrödinger Function

$$\psi(x,t+\varepsilon)=\int_{-\infty}^{\infty}K(x,t+\varepsilon,y,t)\psi(y,t)\mathrm{d}y \tag{120}$$

Dirac's remark 给出，当 $\varepsilon\to 0^+$ 时

$$K(x,t+\varepsilon,y,t)=C\exp\left[\frac{i\varepsilon}{\hbar}L\left(\frac{x+y}{2},\frac{x-y}{\varepsilon},t\right)\right]=C\exp\left\{\frac{i\varepsilon}{\hbar}\left[\frac{m}{2}\left(\frac{x-y}{\varepsilon}\right)^2-V\left(\frac{x+y}{2},t\right)\right]\right\} \tag{121}$$

代回 Eq.(120)

$$\psi(x,t+\varepsilon)=C\int_{-\infty}^{\infty}\exp\left\{\frac{i\varepsilon}{\hbar}\left[\frac{m}{2}\left(\frac{x-y}{\varepsilon}\right)^2-V\left(\frac{x+y}{2},t\right)\right]\right\}\psi(y,t)\mathrm{d}y \tag{122}$$

令 $x=y-\eta$

$$\psi(x,t+\varepsilon)=C\int_{-\infty}^{\infty}\mathrm{d}\eta\exp\left\{\frac{i\varepsilon}{\hbar}\left[\frac{m\eta^2}{2\varepsilon^2}-V\left(x+\frac{\eta}{2},t\right)\right]\right\}\psi(x+\eta,t) \tag{123}$$

展开

$$\psi(x,t)+\varepsilon\frac{\partial}{\partial t}\psi=C\int_{-\infty}^{\infty}\mathrm{d}\eta\exp\left(\frac{im\eta^2}{2\hbar\varepsilon}\right)\exp\left[-\frac{i\varepsilon}{\hbar}V\left(x+\frac{\eta}{2},t\right)\right]\left[\psi(x,t)+\eta\frac{\partial}{\partial x}\psi+\frac{1}{2}\eta^2\frac{\partial^2}{\partial x^2}\psi+\cdots\right] \tag{124}$$

由于

$$\frac{m\eta^2}{2\hbar\varepsilon} \sim O(1) \qquad\qquad \eta \sim O(\sqrt{\varepsilon}) \tag{125}$$

ε 和 η 是小量

$$\psi(x,t) + \varepsilon\frac{\partial}{\partial t}\psi = C\int_{-\infty}^{\infty} \mathrm{d}\eta \exp\left(\frac{im\eta^2}{2\hbar\varepsilon}\right)\left[1 - \frac{i\varepsilon}{\hbar}V(x,t)\right]\left[\psi(x,t) + \eta\frac{\partial}{\partial x}\psi + \frac{1}{2}\eta^2\frac{\partial^2}{\partial x^2}\psi + \cdots\right] \tag{126}$$

已知积分

$$\int_{-\infty}^{\infty} \mathrm{d}\eta \exp\left(\frac{im\eta^2}{2\hbar\varepsilon}\right) = \sqrt{\frac{2\pi\hbar\varepsilon i}{m}} \tag{127}$$

$$\int_{-\infty}^{\infty} \mathrm{d}\eta \exp\left(\frac{im\eta^2}{2\hbar\varepsilon}\right)\eta = 0 \tag{128}$$

$$\int_{-\infty}^{\infty} \mathrm{d}\eta \exp\left(\frac{im\eta^2}{2\hbar\varepsilon}\right)\eta^2 = \frac{\sqrt{\pi}}{2}\left(\frac{2\hbar\varepsilon i}{m}\right)^{\frac{3}{2}} \tag{129}$$

则

$$C\sqrt{\frac{2\pi\hbar\varepsilon i}{m}} = 1 \qquad \Rightarrow \qquad C = \sqrt{\frac{m}{2\pi\hbar\varepsilon i}} \tag{130}$$

$$C\int_{-\infty}^{\infty} \mathrm{d}\eta \exp\left(\frac{im\eta^2}{2\hbar\varepsilon}\right)\eta^2 = \frac{\sqrt{\pi}}{2}\left(\frac{2\hbar\varepsilon i}{m}\right)^{\frac{3}{2}}\sqrt{\frac{m}{2\pi\hbar\varepsilon i}} = \frac{i\hbar\varepsilon}{m} \tag{131}$$

$$\begin{aligned}\psi(x,t) + \varepsilon\frac{\partial}{\partial t}\psi =& C\int_{-\infty}^{\infty} \mathrm{d}\eta \exp\left(\frac{im\eta^2}{2\hbar\varepsilon}\right)\left[1 - \frac{i\varepsilon}{\hbar}V(x,t)\right]\left[\psi(x,t) + \eta\frac{\partial}{\partial x}\psi + \frac{1}{2}\eta^2\frac{\partial^2}{\partial x^2}\psi + \cdots\right] \\ =& \left[1 - \frac{i\varepsilon}{\hbar}V(x,t)\right]\left[\psi(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x,t)\frac{i\hbar\varepsilon}{m}\right]\end{aligned} \tag{132}$$

整理得

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\psi(x,t) \tag{133}$$

10 Formulation in the Phase Space

$$
\begin{aligned}
&K(x,t;x',t') \\
=&\int\prod_{j=1}^{N-1}\mathrm{d}x_j\,\langle x|\exp\left(-\frac{i\varepsilon}{2m\hbar}p^2\right)\exp\left[-\frac{i\varepsilon}{\hbar}V(x)\right]|x_{N-1}\rangle\,\langle x_{N-1}|\exp\left(-\frac{i\varepsilon}{2m\hbar}p^2\right)\exp\left[-\frac{i\varepsilon}{\hbar}V(x)\right]|x_{N-2}\rangle\cdots \\
&\langle x_1|\exp\left(-\frac{i\varepsilon}{2m\hbar}p^2\right)\exp\left[-\frac{i\varepsilon}{\hbar}V(x)\right]|x'\rangle \\
=&\int\prod_{j=1}^{N-1}\mathrm{d}x_j\prod_{i=1}^{N}\mathrm{d}p_i\,\langle x|\exp\left(-\frac{i\varepsilon}{2m\hbar}p^2\right)|p_N\rangle\,\langle p_N|\exp\left[-\frac{i\varepsilon}{\hbar}V(x)\right]|x_{N-1}\rangle\,\langle x_{N-1}|\exp\left(-\frac{i\varepsilon}{2m\hbar}p^2\right)|p_{N-1}\rangle \\
&\langle p_{N-1}|\exp\left[-\frac{i\varepsilon}{\hbar}V(x)\right]|x_{N-2}\rangle\cdots\langle x_1|\exp\left(-\frac{i\varepsilon}{2m\hbar}p^2\right)|p_1\rangle\,\langle p_1|\exp\left[-\frac{i\varepsilon}{\hbar}V(x)\right]|x'\rangle \\
=&\left(\frac{1}{\sqrt{2\pi\hbar}}\right)^{2N}\int\prod_{j=1}^{N-1}\mathrm{d}x_j\prod_{i=1}^{N}\mathrm{d}p_i\exp\left(-\frac{i\varepsilon}{2m\hbar}p_N^2\right)\exp\left(\frac{ip_Nx_N}{\hbar}\right)\exp\left(-\frac{ip_Nx_{N-1}}{\hbar}\right)\exp\left[-\frac{i\varepsilon}{\hbar}V(x_{N-1})\right] \\
&\exp\left(-\frac{i\varepsilon}{2m\hbar}p_{N-1}^2\right)\exp\left(\frac{ip_{N-1}x_{N-1}}{\hbar}\right)\exp\left(-\frac{ip_{N-1}x_{N-2}}{\hbar}\right)\exp\left[-\frac{i\varepsilon}{\hbar}V(x_{N-2})\right]\cdots \\
&\exp\left(-\frac{i\varepsilon}{2m\hbar}p_1^2\right)\exp\left(\frac{ip_1x_1}{\hbar}\right)\exp\left(-\frac{ip_1x'}{\hbar}\right)\exp\left[-\frac{i\varepsilon}{\hbar}V(x')\right] \\
=&\left(\frac{1}{\sqrt{2\pi\hbar}}\right)^{2N}\int\prod_{j=1}^{N-1}\mathrm{d}x_j\prod_{i=1}^{N}\mathrm{d}p_i\exp\left\{\frac{i\varepsilon}{\hbar}\left[p_i\frac{x_i-x_{i-1}}{\varepsilon}-\frac{p_i^2}{2m}-V(x_{i-1})\right]\right\} \\
=&\left(\frac{1}{\sqrt{2\pi\hbar}}\right)^{2N}\int\prod_{j=1}^{N-1}\mathrm{d}x_j\prod_{i=1}^{N}\mathrm{d}p_i\exp\left[\frac{i\varepsilon}{\hbar}\left(p_i\dot{x}-H\right)\right] \\
=&\left(\frac{1}{\sqrt{2\pi\hbar}}\right)^{2N}\int\prod_{j=1}^{N-1}\mathrm{d}x_j\prod_{i=1}^{N}\mathrm{d}p_i\exp\left(\frac{iS}{\hbar}\right)
\end{aligned}
\tag{134}
$$

$$K(x,t;x',t')=\int D[x]D[p]\exp\left(\frac{iS}{\hbar}\right) \tag{135}$$

11 从 Feynman 路径积分导出 $K(x,t;x',t')=\delta(x-x')\quad(t\to t')$

$$K(x,t;x',t')=\int\mathrm{d}x_1\int\mathrm{d}_x\cdots\int\mathrm{d}x_{N-1}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{N}{2}}\exp\left[\frac{im}{2\hbar\varepsilon}\sum_{j=0}^{N-1}(x_{j+1}-x_j)^2\right] \tag{136}$$

$$\lim_{\varepsilon\to0}\exp\left[\frac{im}{2\hbar\varepsilon}(x_{J+1}-x_j)^2\right]=\sqrt{-\frac{2\pi\hbar\varepsilon}{m}}e^{-\frac{i\pi}{4}}\delta(x_{j+1}-x_j) \tag{137}$$

$$
\begin{aligned}
K(x,t;x',t')=&\int\mathrm{d}x_1\int\mathrm{d}_x\cdots\int\mathrm{d}x_{N-1}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{N}{2}}\exp\left[\frac{im}{2\hbar\varepsilon}\sum_{j=0}^{N-1}(x_{j+1}-x_j)^2\right] \\
=&\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{N}{2}}\left(-\frac{2\pi\hbar\varepsilon}{m}\right)^{\frac{N}{2}}\left(e^{-\frac{i\pi}{4}}\right)^{N}\int\mathrm{d}x_1\int\mathrm{d}_x\cdots\int\mathrm{d}x_{N-1}\prod_{j=0}^{N-1}\delta(x_{j+1}-x_j) \\
=&(i)^{\frac{N}{2}}\left(e^{-\frac{i\pi}{2}}\right)^{\frac{N}{2}}\delta(x-x')=\delta(x-x')
\end{aligned}
\tag{138}
$$

12 从 Feynman 路径积分导出 Time-dependent Case 的结论

$$|\psi(t)\rangle = U(t,t')\,|\psi(t')\rangle \tag{139}$$

$$i\hbar\frac{\partial}{\partial t}U(t,t') = H(t)U(t,t') \tag{140}$$

设 $H(t)=H$ 时的解是

$$U(t,t') = \exp\left[-\frac{iH(t-t')}{\hbar}\right] \tag{141}$$

接下来求 $U(t,t')$ 的形式解

$$\mathrm{d}U(t,t') = \frac{1}{i\hbar}H(t)U(t,t')\mathrm{d}t \tag{142}$$

$$\int_{t'}^{t''}\mathrm{d}U(t,t') = \frac{1}{i\hbar}\int_{t'}^{t''}H(t)U(t,t')\mathrm{d}t \tag{143}$$

由于 $U(t',t')=1$，得到

$$U(t'',t') - 1 = \frac{1}{i\hbar}\int_{t'}^{t''}H(t)U(t,t')\mathrm{d}t \tag{144}$$

$$\begin{aligned}
U(t'',t') =& 1 + \frac{1}{i\hbar}\int_{t'}^{t''}H(t_1)U(t_1,t')\mathrm{d}t_1 \\
=& 1 + \frac{1}{i\hbar}\int_{t'}^{t''}H(t_1)\left[1 + \frac{1}{i\hbar}\int_{t'}^{t_1}H(t_2)U(t_2,t')\mathrm{d}t_2\right]\mathrm{d}t_1 \\
=& 1 + \frac{1}{i\hbar}\int_{t'}^{t''}H(t_1)\mathrm{d}t_1 + \left(\frac{1}{i\hbar}\right)^2\int_{t'}^{t''}\mathrm{d}t_1\int_{t'}^{t_2}\mathrm{d}t_2 H(t_1)H(t_2)U(t_2,t') \\
=& \cdots \\
=& 1 + \sum_{n=1}^{\infty}\left(\frac{1}{i\hbar}\right)^N\int_{t'}^{t''}\mathrm{d}t_1\int_{t'}^{t_2}\mathrm{d}t_2\cdots\int_{t'}^{t_{n-1}}\mathrm{d}t_n H(t_1)H(t_2)\cdots H(t_n) \\
=& (t''-t')\exp\left[-\frac{i}{\hbar}\int_{t'}^{t''}\mathrm{d}tH(t)\right]
\end{aligned} \tag{145}$$

故

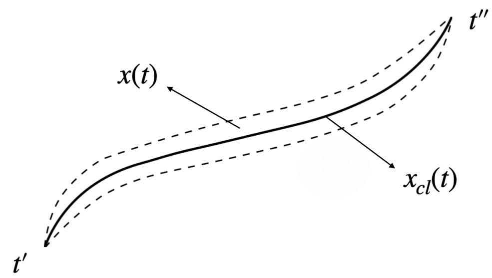
$$U(t,t') = T\exp\left[-\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}t''H(t'')\right] \tag{146}$$

13 General Method for Calculating the Propagator (Semiclassical Method)

$$K(x'',t'';x',t') = \int D[x(t)]e^{\frac{i}{\hbar}S} \tag{147}$$

其中

$$S = \int_{t'}^{t''}\mathrm{d}tL(x,\dot{x},t) = \int_{t'}^{t''}\mathrm{d}t\left[\frac{1}{2}m\dot{x}^2 - V(x,t)\right] \tag{148}$$



$$x(t) = x_{\rm cl}(t) + q(t) \tag{149}$$

其中 $q(t)$ 来源于量子涨落 (QM-Fluctuation)，是小量，$x_{\rm cl}(t)$ 满足

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_{\rm cl}}\right) - \frac{\partial L}{\partial x_{\rm cl}} = 0 \tag{150}$$

$$\dot{x}(t) = \dot{x}_{\rm cl}(t) + \dot{q}(t) \qquad \dot{q}(t') = \dot{q}(t'') \tag{151}$$

$$\begin{aligned} S =& \int_{t'}^{t''} \mathrm{d}t \left[\frac{1}{2} m \left(\dot{x}_{\rm cl} + \dot{q}\right)^2 - V\left(x_{\rm cl} + q, t\right)\right] \\ =& \int_{t'}^{t''} \mathrm{d}t \left[\frac{1}{2} m \left(\dot{x}_{\rm cl}^2 + 2\dot{x}_{\rm cl}\dot{q} + \dot{q}^2\right) - V\left(x_{\rm cl}, t\right) - \frac{\partial V}{\partial x_{\rm cl}} q\right] \\ =& S_{\rm cl} + \int_{t'}^{t''} \mathrm{d}t \left[\frac{1}{2} m \left(2\dot{x}_{\rm cl}\dot{q} + \dot{q}^2\right) - \frac{\partial V}{\partial x_{\rm cl}} q\right] \end{aligned} \tag{152}$$

由于

$$\int_{t'}^{t''} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}q\right) = \int_{t'}^{t''} \mathrm{d}t \left(\ddot{x}q + \dot{x}\dot{q}\right) = 0 \tag{153}$$

则

$$\dot{x}\dot{q} = -\ddot{x}q \tag{154}$$

$$S = S_{\rm cl} + \int_{t'}^{t''} \mathrm{d}t \left[\frac{1}{2} m \left(-2\ddot{x}q + \dot{q}^2\right) - \frac{\partial V}{\partial x_{\rm cl}} q\right] \tag{155}$$

又

$$m\ddot{x}_{\rm cl} = -\frac{\partial V}{\partial x_{\rm cl}} \tag{156}$$

$$S = S_{\rm cl} + \int_{t'}^{t''} \mathrm{d}t \left(\frac{1}{2} m\dot{q}^2 - \frac{1}{2}\frac{\partial^2 V}{\partial x_{\rm cl}^2} q^2\right) = S_{\rm cl} + S_{\text{QM-F}} \tag{157}$$

量子涨落

$$\begin{aligned} S_{\text{QM-F}} =& \int_{t'}^{t''} \mathrm{d}t \left(\frac{1}{2} m\dot{q}^2 - \frac{1}{2}\frac{\partial^2 V}{\partial x_{\rm cl}^2} q^2\right) \\ =& \frac{m}{2}\int_{t'}^{t''} \mathrm{d}t q \left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{1}{m}\frac{\partial^2 V}{\partial x_{\rm cl}^2}\right) q \\ =& \frac{m}{2}\int_{t'}^{t''} \mathrm{d}t q(t) A(t) q(t) \end{aligned} \tag{158}$$

其中

$$A(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{1}{m}\frac{\partial^2 V}{\partial x_{\rm cl}^2} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \omega(t) \tag{159}$$

$$\omega(t) = \frac{1}{m}\frac{\partial^2 V}{\partial x_{\rm cl}^2} \tag{160}$$

$A(t)$ 是厄米算符

$$A(t)\varphi_n(t) = \lambda_n \varphi_n(t) \tag{161}$$

$\varphi_n(t)$ 构成完备基，用 $\varphi_n(t)$ 展开 $q(t)$

$$q(t) = \sum_{n=1}^{\infty} a_n \varphi_n(t) = \lim_{N\to\infty} \sum_{n=1}^{N} a_n \varphi_n(t) \tag{162}$$

$$\begin{aligned}
S_{\text{QM-F}} =& \frac{m}{2} \sum_{n,m} \int_{t'}^{t''} \mathrm{d}t a_n a_m \varphi_n(t) A(t) \varphi_m(t) \\
=& \frac{m}{2} \sum_{n,m} a_n a_m \int_{t'}^{t''} \mathrm{d}t \varphi_n(t) A(t) \varphi_m(t) \\
=& \frac{m}{2} \sum_{n,m} a_n a_m \lambda_m \int_{t'}^{t''} \mathrm{d}t \varphi_n(t) \varphi_m(t) \\
=& \frac{m}{2} \sum_{n,m} a_n a_m \lambda_m \delta_{m,n} T \\
=& \frac{m}{2} \sum_{n} a_n^2 \lambda_n T
\end{aligned} \tag{163}$$

$$\begin{aligned}
K(x'', t''; x', t') =& \int D[x(t)] e^{\frac{i}{\hbar} S} \\
=& e^{\frac{i}{\hbar} S_{\text{cl}}} \int D[q(t)] e^{\frac{i}{\hbar} S_{QM-F}} \\
=& e^{\frac{i}{\hbar} S_{\text{cl}}} \int D[q(t)] \exp\left(\frac{im}{2\hbar} \sum_n a_n^2 \lambda_n T \right) \\
=& e^{\frac{i}{\hbar} S_{\text{cl}}} \pi^N N! \left(\frac{m}{2\pi i \hbar T} \right)^{\frac{N+1}{2}} \int \prod_{n=1}^{N} \mathrm{d}a_n \exp\left(\frac{im}{2\hbar} \sum_n a_n^2 \lambda_n T \right) \\
=& e^{\frac{i}{\hbar} S_{\text{cl}}} \pi^N N! \left(\frac{m}{2\pi i \hbar T} \right)^{\frac{N+1}{2}} \left(\frac{2\pi \hbar i}{mT} \right)^{\frac{N}{2}} \frac{1}{\sqrt{\prod_{n=1}^{N} \lambda_n}} \\
=& e^{\frac{i}{\hbar} S_{\text{cl}}} \pi^N N! \left(\frac{m}{2\pi i \hbar T} \right)^{\frac{1}{2}} T^{-N} \frac{1}{\sqrt{\prod_{n=1}^{N} \lambda_n}}
\end{aligned} \tag{164}$$

Example: 1-D Free Particle

$$\lambda_n = \left(\frac{n\pi}{T} \right)^2 \tag{165}$$

$$\prod_{n=1}^{N} \lambda_n = \prod_{n=1}^{N} \left(\frac{n\pi}{T} \right)^2 = (N!)^2 \frac{\pi^{2N}}{T^{2N}} \tag{166}$$

$$\begin{aligned}
K(x'', t''; x', t') &= e^{\frac{i}{\hbar} S_{\text{cl}}} \pi^N N! \left(\frac{m}{2\pi i \hbar T} \right)^{\frac{1}{2}} T^{-N} \frac{1}{\sqrt{\prod_{n=1}^{N} \lambda_n}} \\
&= e^{\frac{i}{\hbar} S_{\text{cl}}} \pi^N N! \left(\frac{m}{2\pi i \hbar T} \right)^{\frac{1}{2}} T^{-N} \frac{T^N}{N! \pi^N} \\
&= \left(\frac{m}{2\pi i \hbar T} \right)^{\frac{1}{2}} e^{\frac{i}{\hbar} S_{\text{cl}}}
\end{aligned} \tag{167}$$